BORDERED FLOER HOMOLOGY AND EXISTENCE OF INCOMPRESSIBLE TORI IN HOMOLOGY SPHERES

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Abstract. Let K denote a knot inside the homology sphere Y. The zero-framed longitude of K gives the complement of K in Y the structure of a bordered three-manifold, which may be denoted by $Y(K)$. We compute the bordered Floer complex $\widehat{\mathrm{CFD}}(Y(K))$ in terms of the knot Floer complex associated with K. As a corollary, we show that if a homology sphere has the same Heegaard Floer homology as S^3 it does not contain any incompressible tori. Consequently, if Y is an irreducible homology sphere L-space then Y is either S^3, or the Poicaré sphere $\Sigma(2,3,5)$, or it is hyperbolic.

1. Introduction

1.1. **Background and the main results.** Heegaard Floer homology, defined by Ozsváth and Szabó [OS1], has been powerful in extracting topological properties of three-manifolds. Although there is a variety of L-spaces- the three-manifolds with most simple Heegaard Floer homology- in rare cases homology spheres have the Heegaard Floer homology of S^3. The Poincaré sphere $\Sigma(2,3,5)$ is an example of an irreducible homology sphere with $\widehat{\mathrm{HF}}(\Sigma(2,3,5)) = \widehat{\mathrm{HF}}(S^3) = \mathbb{Z}$. It is thus not true in general, that Heegaard Floer homology is capable to distinguish S^3 from other homology spheres. However, a conjecture of Ozsváth and Szabó predicts that the $\Sigma(2,3,5)$ is the only non-trivial example of an irreducible homology sphere with trivial Heegaard Floer homology. In this paper, we will address the case of a 3-manifold which contains an incompressible torus. Let $\mathbb{F}$ denote the field $\mathbb{Z}/2\mathbb{Z}$ throughout this paper.

Theorem 1.1. *If a homology sphere Y contains an incompressible torus*

$$\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F} = \widehat{\mathrm{HF}}(S^3;\mathbb{F}).$$

Together with Thurston's geometrization conjecture, now a theorem of Perelman (see [Thu, Per], also [MT1, MT2]), this result reduces the study of Ozsváth-Szabó conjecture to the homology spheres which are either Seifert fibered or hyperbolic. It may be shown (see [Rus], also [Ef5]) that Poincaré sphere and the standard sphere are the only Seifert fibered homology spheres with trivial Heegaard Floer homology. Thus, Ozsváth-Szabó conjecture is reduced to the following.

Conjecture 1.2. *If the homology sphere Y is hyperbolic* $\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F}$.

The reduced Khovanov homology of a knot K inside the standard three-sphere is related to the Heegaard Floer homology of the double cover of the standard sphere branched over K [OS4]. Ozsváth-Szabó conjecture (or Conjecture 1.2) thus implies

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that the reduced Khovanov homology (and thus Khovanov homology) detects the unknot; a theorem of Kronheimer and Mrowka [KM]. The result of this paper reproves a few special cases of the aforementioned theorem. A knot $K \subset S^3$ is π-*hyperbolic* if $S^3 - K$ admits a Riemannian metric with constant negative curvature which becomes singular folding with an angle π around K.

Corollary 1.3. *Suppose that for a knot $K \subset S^3$ one of the following is true:*

- *K is not π-hyperbolic.*
- *K is a prime satellite knot.*

Then the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ is greater than 1.

1.2. **Bordered Floer homology for a knot complement.** The proof of Theorem 1.1 rests heavily on a construction of the bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ associated with the complement of a knot $K \subset Y$ using the knot Floer complex $\mathrm{CFK}(Y,K)$. Consider a doubly pointed Heegaard diagram $(\Sigma,\alpha,\beta;u,v)$ for K. The markings u and v give the map

$$\mathfrak{s} = \mathfrak{s}_{u,v} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\mathrm{Spin}}^c(Y,K)$$

where $\mathfrak{s}(\mathbf{x})$ denotes the relative Spin^c class assigned to $\mathbf{x}$ in the sense of [Ni], which is defined by assigning a nowhere vanishing vector field on $Y - \mathrm{nd}(K)$ to $\mathbf{x}$ which is tangent to the boundary. Multiplying the vector fields by -1 gives a map

$$J : \underline{\mathrm{Spin}}^c(Y,K) \longrightarrow \underline{\mathrm{Spin}}^c(Y,K)$$

and the map $\mathfrak{s} \mapsto c_1(\mathfrak{s}) = \mathfrak{s} - J(\mathfrak{s}) \in \mathrm{H}^2(Y,K;\mathbb{Z})$ gives an identification of $\underline{\mathrm{Spin}}^c(Y,K)$ with $\mathbb{Z}$, which will be implicit throughout this paper. Let

$$C = \big\langle [\mathbf{x},i,j] \;\big|\; \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ \mathfrak{s}(\mathbf{x}) - i + j = 0 \big\rangle_{\mathbb{Z}}$$

denote the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex associated with K. Following [OS3] we may consider the sub-modules

$$C\{i=a, j=b\}, C\{i=a, j\le b\} \;\text{ and }\; C\{i \le a, j=b\} \qquad a,b \in \mathbb{Z}\cup\{\infty\}$$

with the induced structure as a chain complex. Set $C\{i=a\}$ to be the chain complex $C\{i=a, j\le\infty\}$ and $C\{j=b\}$ to be $C\{i\le\infty, j=b\}$. For any relative Spin^c class $\mathfrak{s} \in \mathbb{Z} = \underline{\mathrm{Spin}}^c(Y,K)$ let

$$\begin{aligned} &i_n^{\mathfrak{s}} = i_n^{\mathfrak{s}}(K) : C\{i \le \mathfrak{s}, j=0\} \oplus C\{i=0, j \le n-\mathfrak{s}-1\} \longrightarrow C\{j=0\}\\ &i_n^{\mathfrak{s}}\left([\mathbf{x},i,0],[\mathbf{y},0,j]\right) := [\mathbf{x},i,0] + \Xi[\mathbf{y},0,j], \end{aligned}$$

where $\Xi : C\{i=0\} \to C\{j=0\}$ is the chain homotopy equivalence corresponding to the Heegaard moves which change the diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$ to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};v)$. Let $Y_n(K)$ denote the three-manifold obtained from Y by n-surgery on K and let K_n denote the corresponding knot inside $Y_n(K)$ which is determined by the aforementioned surgery.

Proposition 1.4. *The homology of the mapping cone $M(i_n^{\mathfrak{s}})$ gives*

$$\mathbb{H}_n(K,\mathfrak{s}) = \widehat{\mathrm{HFK}}(Y_n(K), K_n, \mathfrak{s}).$$

Note that $M(i_0^{\mathfrak{s}})$ is a sub-complex of both $M(i_1^{\mathfrak{s}})$ and $M(i_1^{\mathfrak{s}+1})$. We denote the embedding maps by $F_\infty^{\mathfrak{s}} = F_\infty^{\mathfrak{s}}(K)$ and $\overline{F}_\infty^{\mathfrak{s}+1} = \overline{F}_\infty^{\mathfrak{s}+1}(K)$, respectively. The quotient of $M(i_1^{\mathfrak{s}})$ by $F_\infty^{\mathfrak{s}}\left(M(i_0^{\mathfrak{s}})\right)$ is isomorphic to

$$\widehat{\mathrm{CFK}}(K,\mathfrak{s}) \simeq C\{i=0, j=-\mathfrak{s}\}.$$

Denote the quotient map by $F_0^{\mathfrak{s}} = F_0^{\mathfrak{s}}(K)$. Similarly, define the quotient map $\overline{F}_0^{\mathfrak{s}} = \overline{F}_0^{\mathfrak{s}}(K)$ from $M(i_1^{\mathfrak{s}})$ to $M(i_1^{\mathfrak{s}})/\mathrm{Im}(\overline{F}_\infty^{\mathfrak{s}})$. The short exact sequences

$$0 \longrightarrow M\left(i_0^{\mathfrak{s}}\right) \xrightarrow{F_\infty^{\mathfrak{s}}} M\left(i_1^{\mathfrak{s}}\right) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow M\left(i_0^{\mathfrak{s}-1}\right) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}}} M\left(i_1^{\mathfrak{s}}\right) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0$$

give the following two homology exact triangles

$$\begin{tikzcd}
\mathbb{H}_0(\mathfrak{s}) \arrow[rr, "\mathfrak{f}_\infty^{\mathfrak{s}}"] & & \mathbb{H}_1(\mathfrak{s}) \arrow[dl, "\mathfrak{f}_0^{\mathfrak{s}}"] \\
& \mathbb{H}_\infty(\mathfrak{s}) \arrow[ul, "\mathfrak{f}_1^{\mathfrak{s}}"] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{H}_0(\mathfrak{s}-1) \arrow[rr, "\overline{\mathfrak{f}}_\infty^{\mathfrak{s}}"] & & \mathbb{H}_1(\mathfrak{s}) \arrow[dl, "\overline{\mathfrak{f}}_0^{\mathfrak{s}}"] \\
& \mathbb{H}_\infty(\mathfrak{s}) \arrow[ul, "\overline{\mathfrak{f}}_1^{\mathfrak{s}}"] &
\end{tikzcd}, \tag{1}$$

where $\mathbb{H}_\bullet(\mathfrak{s}) = \mathbb{H}_\bullet(K,\mathfrak{s})$. We let

$$C_\bullet(K) = \bigoplus_{\mathfrak{s}\in\mathbb{Z}} C_\bullet(K,\mathfrak{s}) \;\; \text{and} \;\; \mathbb{H}_\bullet(K) = \bigoplus_{\mathfrak{s}\in\mathbb{Z}} \mathbb{H}_\bullet(K,\mathfrak{s}), \quad \bullet \in \{0,1,\infty\}$$

where $C_\bullet(K,\mathfrak{s}) = M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$ and $C_\infty(K,\mathfrak{s}) = C\{i=\mathfrak{s}, j=0\}$. Denote the differential of $C_\bullet(K)$ by $d_\bullet$ for $\bullet \in \{0,1,\infty\}$. Set $M(K) = C_0(K) \oplus C_1(K)$ and $L(K) = C_1(K) \oplus C_\infty(K)$. Let $F_\bullet = F_\bullet(K)$ denote the map obtained by putting all $F_\bullet^{\mathfrak{s}}$ together. These maps will be called the *bypass homomorphisms*.

The zero framed longitude of K and its meridian give a parametrization of the the torus boundary $-T^2$ of $Y \setminus \mathrm{nd}(K)$. The corresponding bordered three-manifold is denoted by $Y(K)$. A differential graded algebra $\mathcal{A}(T^2,0)$ is associated with T^2. The bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ is then a module over the differential graded algebra $\mathcal{A}(T^2,0)$. Following the notation of Subsection 4.2 from [LOT2], $\mathcal{A}(T^2,0)$ is generated, as a module over $\mathbb{F}$, by the idempotents $\imath_0$ and $\imath_1$, and the chords $\rho_1,\rho_2,\rho_3,\rho_{12}=\rho_1\rho_2, \rho_{23}=\rho_2\rho_3$ and $\rho_{123}=\rho_1\rho_2\rho_3$;

$$\mathcal{A}(T^2,0) = \Big\langle \begin{tikzcd} \imath_0 \bullet \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_3"'] & \bullet\, \imath_1 \arrow[l, "\rho_2"'] \end{tikzcd} \Big\rangle \Big/ \left(\rho_2\rho_1 = \rho_3\rho_2 = 0\right).$$

Theorem 1.5. *With the above notation, the bordered Floer complex $\widehat{\mathrm{CFD}}(Y(K))$ is quasi-isomorphic to the left module over the differential graded algebra $\mathcal{A}(T^2,0)$, which is generated by $\imath_0.L(K)$ and $\imath_1.M(K)$ and is equipped with the differential $\partial\colon \widehat{\mathrm{CFD}}(Y(K)) \to \widehat{\mathrm{CFD}}(Y(K))$ defined by*

$$\partial\binom{\mathbf{x}}{\mathbf{y}} = \begin{cases} \begin{pmatrix} d_0(\mathbf{x}) \\ \overline{F}_\infty(\mathbf{x}) + d_1(\mathbf{y}) \end{pmatrix} + \rho_2.\binom{0}{\mathbf{x}} & \text{if } \binom{\mathbf{x}}{\mathbf{y}} \in M(K) \\[2ex] \begin{pmatrix} d_1(\mathbf{x}) \\ F_0(\mathbf{x}) + d_\infty(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \rho_1 F_\infty(\mathbf{x}) \\ \rho_3\overline{F}_0(\mathbf{y}) + \rho_{123}\overline{F}_0(F_\infty(\mathbf{x})) \end{pmatrix} & \text{if } \binom{\mathbf{x}}{\mathbf{y}} \in L(K) \end{cases} \tag{2}$$

2. Surgery on null-homologous knots

2.1. A triangle of chain maps. By a Heegaard n-tuple we mean the data

$$(\Sigma, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n; u_1, ..., u_r)$$

where Σ is a Riemann surface of genus g, each $\boldsymbol{\alpha}_i$ is g-tuples of disjoint simple closed curves for $i = 1, ..., n$ and u_j are markings in $\Sigma - \sqcup_{i=1}^n \boldsymbol{\alpha}_i$. Let $\mathbb{T}_{\alpha_i} \subset \mathrm{Sym}^g(\Sigma)$ denote the torus associated with $\boldsymbol{\alpha}_i$ and let $\mathbf{x}_i \in \mathbb{T}_{\alpha_i} \cap \mathbb{T}_{\alpha_{i+1}}$ for $i = 1, ..., n-1$ and $\mathbf{x}_n \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\alpha_n}$ be n intersection points. Let $\pi_2(\mathbf{x}_1, ..., \mathbf{x}_n)$ denote the set of homotopy classes of n-gons connecting $\mathbf{x}_1, ..., \mathbf{x}_n$ and define $\pi_2^j(\mathbf{x}_1, ..., \mathbf{x}_n; u_1, ..., u_r)$ to be the subset of $\pi_2(\mathbf{x}_1, ..., \mathbf{x}_n)$ which consists of the classes with Maslov index j which have zero intersection number with the codimension one sub-varieties $L_{u_1}, ..., L_{u_r}$ of $\mathrm{Sym}^g(\Sigma)$ which correspond to the markings $u_1, ..., u_r$.

Let $K \subset Y$ be a knot inside a homology sphere Y. Consider a Heegaard diagram

$$H = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}, p_\infty)$$

for the pair (Y, K), where β_g corresponds to the meridian of K and the marking p_∞ is placed on β_g, so that putting a pair of marked points near p_∞ and on the two sides of β_g we obtain a doubly pointed Heegaard diagram for K. Suppose that λ represents a zero-framed longitude for the knot K. Let λ_n be a small perturbation of the juxtaposition $\lambda + n\beta_g$ and β_i^n denote a small Hamiltonian isotope of β_i for $i = 1, ..., g-1$. The Heegaard diagram

$$H_n = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n = \{\beta_1^n, ..., \beta_{g-1}^n, \lambda_n\}, p_n)$$

gives a diagram for $(Y_n(K), K_n)$, where p_n is a marked point at the intersection of λ_n and β_g. With the integers $m > n \geq 0$ fixed, we assume that λ_n and λ_{n+m} intersect each other in m transverse points, and that for an intersection point q of these latter curves the points q, p_n, p_{m+n} are the vertices of a triangle Δ, which is one of the connected components in $\Sigma - (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \{\lambda_n, \lambda_{n+m}\})$. From the 4 quadrants which have q as a corner two of them belong to the neighbors of Δ. Place a pair of markings u and v in these two quadrants, and use them as the punctures in the following discussion.

Fix a relative Spin^c class $\mathfrak{s} \in \underline{\mathrm{Spin}^c}(Y, K) = \mathbb{Z}$. The complex associated with the Heegaard diagram $R_n = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v)$ and the relative Spin^c class $\mathfrak{s}$ is denoted by $\widehat{\mathrm{CFK}}(K_n, \mathfrak{s})$, while the complex associated with the Heegaard diagram $R_{n+m} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$ and the relative Spin^c classes $\mathfrak{s}$ and $\mathfrak{s} + m$ is denoted by

$$\widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s} + m).$$

Let Θ_f denote the top generator of the Heegaard Floer homology group associated with $(\Sigma, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}; u, v)$. Consider the holomorphic triangle map

$$\begin{aligned} &f^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s} + m) \to \widehat{\mathrm{CF}}(Y) \\ &f^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_f, \mathbf{z}; u, v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}. \end{aligned} \tag{3}$$

The diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$ determines a cobordism from $Y_n(K) \coprod L$ to $Y_{n+m}(K)$, where $L = L(m, 1)\#(\#^{g-1} S^1 \times S^2)$. The intersection point q determines a canonical Spin^c class $\mathfrak{s}_q \in \mathrm{Spin}^c(L)$ in the sense of Definition 3.2 of [OS3]. Let

Θ_g denote the top generator of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m};u,v)$ which corresponds to $\mathfrak{s}_q$, or equivalently to the intersection point q. Define

$$\begin{aligned} g^{\mathfrak{s}} &: \widehat{\mathrm{CFK}}(K_n,\mathfrak{s}) \longrightarrow \widehat{\mathrm{CFK}}(K_{n+m}) \\ g^{\mathfrak{s}}(\mathbf{x}) &:= \sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{n+m}}} \ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_g,\mathbf{z};u,v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}. \end{aligned}$$

Following Section 8 of [AE] (Lemma 8.2 and the discussion after that), if $\mathfrak{s}(\mathbf{x})=\mathfrak{s}$

$$g^{\mathfrak{s}}(\mathbf{x}) \in \widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{n+m},m+\mathfrak{s}).$$

Finally, the top generator $\Theta_h \in \widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}_n;u,v)$ and the Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_n;u,v)$ determine the map $h^{\mathfrak{s}}$ on $\widehat{\mathrm{CF}}(Y)$, which is defined by

$$h^{\mathfrak{s}}(\mathbf{x}) := \sum_{\substack{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}\\ \mathfrak{s}(\mathbf{z})=\mathfrak{s}}} \ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_h,\mathbf{z};u,v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}.$$

We thus arrive at the triangle of chain maps

$$\begin{tikzcd} \widehat{\mathrm{CF}}(Y) \arrow[rr, "h^{\mathfrak{s}}=h^{\mathfrak{s}}_n"] & & \widehat{\mathrm{CFK}}(K_n,\mathfrak{s}) \arrow[dl, "g^{\mathfrak{s}}=g^{\mathfrak{s}}_n"] \\ & \widehat{\mathrm{CFK}}(K_{m+n},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{m+n},\mathfrak{s}+m) \arrow[ul, "f^{\mathfrak{s}}=f^{\mathfrak{s}}_n"] & \end{tikzcd} \tag{4}$$

2.2. **Exactness of triangle.** Let $M(f^{\mathfrak{s}}_n)$ denote the mapping cone of $f^{\mathfrak{s}}=f^{\mathfrak{s}}_n$.

Theorem 2.1. *If m is sufficiently large there is a map*

$$H^{\mathfrak{s}}_{h_n}: \widehat{\mathrm{CFK}}(K_n,\mathfrak{s}) \longrightarrow \widehat{\mathrm{CF}}(Y)$$

which satisfies $d\circ H^{\mathfrak{s}}_{h_n}+H^{\mathfrak{s}}_{h_n}\circ d = f^{\mathfrak{s}}_n\circ g^{\mathfrak{s}}_n$, such that the chain map

$$\begin{aligned} \imath^{\mathfrak{s}}_n &: \widehat{\mathrm{CFK}}(K_n,\mathfrak{s}) \longrightarrow M(f^{\mathfrak{s}}_n) \\ \imath^{\mathfrak{s}}_n(\mathbf{x}) &:= (g^{\mathfrak{s}}_n(\mathbf{x}), H^{\mathfrak{s}}_{h_n}(\mathbf{x})), \quad \forall\ \mathbf{x}\in\widehat{\mathrm{CFK}}(K_n,\mathfrak{s}), \end{aligned}$$

is a quasi-isomorphism.

Proof. The proof is almost identical to the proof used in Section 8 from [AE]. We outline the proof to set up the notation. Let us first define

$$\begin{aligned} H^{\mathfrak{s}}_f &: \widehat{\mathrm{CF}}(Y) \to \widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{n+m},m+\mathfrak{s}) \\ H^{\mathfrak{s}}_f(\mathbf{x}) &:= \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{n+m}}\\ \mathfrak{s}(\mathbf{y})-\mathfrak{s}\equiv 0 \pmod m}} \ \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_h,\Theta_g,\mathbf{y};u,v)} \#\big(\mathcal{M}(\square)\big).\mathbf{y}. \end{aligned}$$

The condition $\mathfrak{s}(\mathbf{y})-\mathfrak{s}\equiv \pmod m$ implies $\mathfrak{s}(\mathbf{y})\in\{\mathfrak{s},\mathfrak{s}+m\}$ since m is large. Considering all possible boundary degenerations of the one-dimensional moduli space corresponding to a square class $\square\in\pi_2^0(\mathbf{x},\Theta_h,\Theta_g,\mathbf{y};u,v)$ we find

$$d\circ H^{\mathfrak{s}}_f + H^{\mathfrak{s}}_f\circ d = h^{\mathfrak{s}}\circ g^{\mathfrak{s}}. \tag{5}$$

For (5) note that the holomorphic triangles $\Delta\in\pi_2^0(\Theta_h,\Theta_g,\Theta;u,v)$ with $\Theta\in\mathbb{T}_{\beta_{n+m}}\cap\mathbb{T}_\beta$ come in canceling pairs, where the difference between the coefficients

of every canceling pair at a marking s (placed on the left-hand-side of β_g and close to u) is always a multiple of m. Similarly, define

$$H_g^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{n+m},m+\mathfrak{s})\longrightarrow\widehat{\mathrm{CF}}(Y),$$
$$H_g^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}}\sum_{\substack{\square\in\pi_2^{-1}(\mathbf{x},\Theta_f,\Theta_h,\mathbf{y};u,v)\\ n_s(\square)\equiv 0\pmod m}}\#\big(\mathcal{M}(\square)\big).\mathbf{y}.$$

Since the contributing holomorphic triangles corresponding to the Heegaard triple $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n;u,v)$ and the closed top generators Θ_f,Θ_h come in canceling pairs,

$$d\circ H_g^{\mathfrak{s}}+H_g^{\mathfrak{s}}=h^{\mathfrak{s}}\circ f^{\mathfrak{s}}. \tag{6}$$

Finally, define the homotopy map $H_h^{\mathfrak{s}}$ by

$$H_h^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})\longrightarrow\widehat{\mathrm{CF}}(Y)$$
$$H_h^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_g,\Theta_f,\mathbf{y};u,v)}\#\big(\mathcal{M}(\square)\big).\mathbf{y}.$$

Employ the same argument again to show that

$$d\circ H_h^{\mathfrak{s}}+H_h^{\mathfrak{s}}\circ d=f^{\mathfrak{s}}\circ g^{\mathfrak{s}}. \tag{7}$$

We next introduce the pentagon maps. Let $\boldsymbol{\beta}_n'$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_n$. Choosing the Hamiltonian isotopy sufficiently small allows us to assume that the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n';u,v)$ and the Spin^c class $\mathfrak{s}$ may be identified with $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$, since the intersection points and the corresponding moduli spaces connecting them change continuously by slight Hamiltonian perturbation of the Lagrangian sub-manifolds. There is a top generator corresponding to $(\boldsymbol{\beta},\boldsymbol{\beta}_n')$ which is in correspondence with Θ_h. We denote this generator by Θ_h'. Define

$$P_f^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K_n,\mathfrak{s}),$$
$$P_f^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}}\sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_g,\Theta_f,\Theta_h',\mathbf{y};u,v)\\ n_s(\pentagon)\equiv 0\pmod m}}\#\big(\mathcal{M}(\pentagon)\big).\mathbf{y},$$

Consider different boundary degenerations of the 1-dimensional moduli space associated with a pentagon of Maslov index -1, and note that

- There is a unique contributing square classes $\square\in\pi_2^{-1}(\Theta_g,\Theta_f,\Theta_h',\Theta)$ of index -1 which corresponds to the quadruple $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n';u,v)$. Moreover, $\Theta=\Theta_n$ is the top generator for the diagram $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_n';u,v)$.
- The contributing triangle classes
$$\Delta\in\pi_2^0(\Theta_g,\Theta_f,\Theta;u,v)\quad\text{and}\quad\Delta'\in\pi_2^0(\Theta_f,\Theta_h',\Theta;u,v)$$
corresponding to the triples $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta};u,v)$ and $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n';u,v)$ come in canceling pairs.

These observations, combined with our earlier arguments, imply that

$$\begin{aligned}&d\circ P_f^{\mathfrak{s}}+P_f^{\mathfrak{s}}\circ d+J_f^{\mathfrak{s}}=h^{\mathfrak{s}}\circ H_h^{\mathfrak{s}}-H_g^{\mathfrak{s}}\circ g^{\mathfrak{s}},\quad\text{where}\\ &J_f^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n'}}\sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_n,\mathbf{y};u,v)}\#\big(\mathcal{M}(\Delta)\big).\mathbf{y}.\end{aligned} \tag{8}$$

Consider the 5-tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}';u,v,z)$, where $\boldsymbol{\beta}'=\{\beta'_1,...,\beta'_g\}$ is a set of g simple closed curves which are obtained from $\boldsymbol{\beta}$ by a small Hamiltonian isotopy. Thus β_i and β'_i intersect each other is a pair of canceling intersection points. We assume that the small area bounded between the two curves β_g and β'_g is formed as a union of two bigons; a small bigon which is a subset of the connected component of $\Sigma^\circ=\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}-\boldsymbol{\beta}_n-\boldsymbol{\beta}_{n+m}$ which contains the marking v and a long and thin bigon which is stretched along β_g. We assume that the marking z is chosen in the intersection of the second bigon with the connected component in Σ° which corresponds to u. If the Hamiltonian perturbation is sufficiently small the chain complex $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u)$ may be identified with $\widehat{\mathrm{CF}}(Y)$. Define

$$\begin{aligned}&P_g^{\mathfrak{s}}:\widehat{\mathrm{CF}}(Y)\longrightarrow\widehat{\mathrm{CF}}(Y)\\&P_g^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'}}\ \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_h,\Theta_g,\Theta'_f,\mathbf{y};u,v)\\ n_z(\pentagon)+\mathfrak{s}(\mathbf{x})\equiv\mathfrak{s}\ (\mathrm{mod}\ m)}}\#\big(\mathcal{M}(\pentagon)\big).\mathbf{y}.\end{aligned}$$

Five types of the ten possible degenerations in the boundary of the 1-dimensional moduli space associated with a pentagon class

$$\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_h,\Theta_g^q,\Theta'_f,\mathbf{y};u,v)\ \ \text{with}\ \ n_z(\pentagon)+\mathfrak{s}(\mathbf{x})\equiv\mathfrak{s}\ (\mathrm{mod}\ m),$$

corresponding to a degeneration to a bigon and a pentagon, contribute to the coefficient of $\mathbf{y}$ in $(d\circ P_g^{\mathfrak{s}}+P_g^{\mathfrak{s}}\circ d)(\mathbf{x})$. The remaining five types correspond to the degenerations of $\pentagon$ into a square and a triangle. The choice of the markings implies that two of these degeneration types contribute to the coefficient of $\mathbf{y}$ in $(f^{\mathfrak{s}}\circ H_f^{\mathfrak{s}}-H_h^{\mathfrak{s}}\circ h^{\mathfrak{s}})(\mathbf{x})$. There is a unique contributing square class, corresponding to $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}';u,v)$ and the intersection points $\Theta_h,\Theta_g,\Theta'_f,\Theta_\infty$, where Θ_∞ denotes the top generator for $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}';u,v)$. Moreover, the triangles which contribute in $\pi_2(\Theta_h,\Theta_g,\Theta_f)$ and $\pi_2(\Theta_g,\Theta'_f,\Theta'_h)$ come in canceling pairs. Thus

$$\begin{aligned}&d\circ P_g^{\mathfrak{s}}+P_g^{\mathfrak{s}}\circ d+J_g^{\mathfrak{s}}=f^{\mathfrak{s}}\circ H_f^{\mathfrak{s}}-H_h^{\mathfrak{s}}\circ h^{\mathfrak{s}},\quad\text{where}\\&J_g^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'}}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_\infty,\mathbf{y};u,v)}\#\big(\mathcal{M}(\Delta)\big).\mathbf{y}\end{aligned}\tag{9}$$

Let $\boldsymbol{\beta}'_{n+m}$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_{n+m}$. Again, we assume that the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}'_{n+m};u,v)$ and the Spin^c classes $\mathfrak{s},\mathfrak{s}+m$ is identified with $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)$. There is a top generator Θ'_g for $(\boldsymbol{\beta}_n,\boldsymbol{\beta}'_{n+m})$ which is in correspondence with Θ_g. Define

$$\begin{aligned}&P_h^{\mathfrak{s}}:\bigoplus_{\mathfrak{t}\in\{\mathfrak{s},\mathfrak{s}+m\}}\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{t})\longrightarrow\bigoplus_{\mathfrak{t}\in\{\mathfrak{s},\mathfrak{s}+m\}}\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{t})\\&P_h^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}}\ \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_f,\Theta_h,\Theta'_g,\mathbf{y};u,v)\\ n_z(\pentagon)\equiv 0\ (\mathrm{mod}\ m)}}\#\big(\mathcal{M}(\pentagon)\big).\mathbf{y}.\end{aligned}$$

A similar argument implies that

$$\begin{aligned}&d\circ P_h^{\mathfrak{s}}+P_h^{\mathfrak{s}}\circ d+J_h^{\mathfrak{s}}=g^{\mathfrak{s}}\circ H_g^{\mathfrak{s}}-H_f^{\mathfrak{s}}\circ f^{\mathfrak{s}},\quad\text{where}\\&J_h^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_{n+m},\mathbf{y};u,v)}\#\big(\mathcal{M}(\Delta)\big).\mathbf{y},\end{aligned}\tag{10}$$

and Θ_{n+m} is the top generator of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}'_{n+m};u,v)$. Since $J^{\mathfrak{s}}_f, J^{\mathfrak{s}}_g, J^{\mathfrak{s}}_h$ are quasi-isomorphisms, Lemma 3.3 from [AE] completes the proof. $\square$

Choose the markings s and t on the Heegaard diagram so that for each one of the pairs (z,s) and (v,t) there is an arc connecting them on the Heegaard surface which cuts β_g in a single transverse point and stays disjoint from all other curves in $\boldsymbol{\alpha}\cup\boldsymbol{\beta}\cup\boldsymbol{\beta}'\cup\boldsymbol{\beta}_n\cup\boldsymbol{\beta}_{n+m}$, see Figure 1. Consider the chain map

$$\begin{aligned}&\Xi\circ\overline{f}^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)\longrightarrow\widehat{\mathrm{CF}}(Y),\\ &\overline{f}^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\mathbf{y};s,t)}\#(\mathcal{M}(\Delta)).\mathbf{y},\end{aligned}$$

where $\Xi:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$ is the chain homotopy equivalence given by the Heegaard moves which change $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$ to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$.

Lemma 2.2. *The chain maps $f^{\mathfrak{s}}$ and $\Xi\circ\overline{f}^{\mathfrak{s}}$ are chain homotopic.*

Proof. Note that the aforementioned Heegaard moves consist of $2g-2$ handle slides (composed with isotopies) on $\boldsymbol{\beta}$, supported away from the markings s,t. Denote the corresponding g-tuples of curves by $\boldsymbol{\beta}^0=\boldsymbol{\beta},\boldsymbol{\beta}^1,...,\boldsymbol{\beta}^{2g-2}$, where $\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{2g-2};s)$ may be identified with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$.
The Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s)$ and the top generator Θ^i of $(\Sigma,\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s,t)$ determine a chain map

$$\Xi^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1};s)\longrightarrow\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;s).$$

The Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;s,t)$ together with the top generator Θ^i_f of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;s,t)$ determines a chain map

$$f^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};s,t)\longrightarrow\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;s).$$

Finally, the Heegaard quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s,t)$ together with Θ^{i-1}_f and Θ^i, determines a homomorphism

$$H^i:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};s,t)\longrightarrow\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^i;s).$$

Considering different boundary degenerations of the one-dimensional moduli space associated with a square class of index 0 we find

$$d\circ H^i+H^i\circ d=f^i+\Xi^i\circ f^{i-1},\qquad i=1,...,2g-2. \tag{11}$$

Let us define

$$H=H^{2g-2}+\Xi^{2g-2}\circ H^{2g-3}+\Xi^{2g-2}\circ\Xi^{2g-3}H^{2g-4}+\dots+(\Xi^{2g-2}\circ\dots\Xi^2)\circ H^1.$$

Using (11) we find

$$d\circ H+H\circ d=f^{2g-2}+\Xi\circ f^0,\quad \text{where } \Xi=\Xi^{2g-2}\circ\dots\Xi^1.$$

Restricting the above equation to $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)$ we are done, once we note that $f^{\mathfrak{s}}$ is the restriction of f^{2g-2} and $\overline{f}^{\mathfrak{s}}$ is the restriction of f^0. $\square$

3. The homomorphisms in the surgery triangle

Consider the triply punctured Heegaard 5-tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0,\boldsymbol{\beta}_1,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)$, as before, and assume that the local picture around the curves $\lambda_0,\lambda_1,\lambda_m,\lambda_\infty$ is the one illustrated in Figure 1. The top generators $\Theta_{0,1}$, Θ_{g_1} and Θ_{f_1} of the Heegaard diagrams $(\Sigma,\boldsymbol{\beta}_0,\boldsymbol{\beta}_1;u,v,w)$, $(\Sigma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_m;u,v,w)$ and $(\Sigma,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)$ (respectively) determine the holomorphic pentagon map

$$\begin{aligned} &P^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\longrightarrow \widehat{\mathrm{CF}}(Y)\\ &P^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\mathbf{y};u,v,w)}\#\left(\mathcal{M}(\pentagon)\right).\mathbf{y}\end{aligned}$$

Every pentagon class $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\mathbf{y};u,v,w)$ corresponds to a 1-dimensional moduli space with boundary. The boundary points are in correspondence with the degeneration of the domain of $\pentagon$ into two parts. Since the generators $\Theta_{0,1},\Theta_{g_1}$ and Θ_{f_1} are closed, the degenerations into a bi-gon and a pentagon correspond to the the coefficient of $\mathbf{y}$ in $(d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d)(\mathbf{x})$. The remaining degenerations are the degenerations $\pentagon=\square\star\Delta$ to a triangle Δ with Maslov index 0 and a square $\square$ with Maslov index -1 which miss u,v and w. The possibilities are

1. $\square\in\pi_2(\mathbf{z},\Theta_{g_1},\Theta_{f_1},\mathbf{y})$ and $\Delta\in\pi_2(\mathbf{x},\Theta_{0,1},\mathbf{z})$,
2. $\square\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z})$ and $\Delta\in\pi_2(\mathbf{z},\Theta_{f_1},\mathbf{y})$,
3. $\square\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta,\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{g_1},\Theta_{f_1},\Theta)$,
4. $\square\in\pi_2(\mathbf{x},\Theta,\Theta_{f_1},\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta)$,
5. $\square\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\Theta)$ and $\Delta\in\pi_2(\mathbf{x},\Theta,\mathbf{y})$.

First type degenerations correspond to the coefficient of $\mathbf{y}$ in $(H^{\mathfrak{s}}_{h_1}\circ\mathfrak{f}^{\mathfrak{s}}_\infty)(\mathbf{x})$, where

$$\mathfrak{f}^{\mathfrak{s}}_\infty:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K_1,\mathfrak{s})$$

is a chain map so that the induced map $\mathfrak{f}^{\mathfrak{s}}_\infty:\mathbb{H}_0(K,\mathfrak{s})\to\mathbb{H}_1(K,\mathfrak{s})$ happens to be the homomorphism which appears in the splicing formula of [Ef4]. Degenerations of type 2 correspond to the coefficient of $\mathbf{y}$ in $f^{\mathfrak{s}}_1\circ H^{\mathfrak{s}}(\mathbf{x})$, where $H^{\mathfrak{s}}$ is defined by

$$\begin{aligned} &H^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K_m,\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)\\ &H^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}}\ \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z};u,v,w)}\#\left(\mathcal{M}(\square)\right).\mathbf{z}.\end{aligned}$$

In a degeneration of type 3, the contributing triangle classes Δ come in canceling pairs. The total (signed) count of such degenerations is thus trivial. Furthermore, there are no holomorphic representatives for the square classes which appear in the boundary degenerations of type 5, i.e. we may assume that there are no such degenerations. In a degeneration of type 4, the moduli space corresponding to Δ is trivial unless $\Theta=\Theta_{g_0}$ and Δ corresponds to the union of small triangles connecting $\Theta_{0,1},\Theta_{g_1}$ and Θ_{g_0}. In this latter case the signed contribution of such triangles is 1. The signed count of such boundary degenerations is thus equal to

$$\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_{g_0},\Theta_{f_1},\mathbf{y};u,v,w)}\#\left(\mathcal{M}(\square)\right).\mathbf{y}=H^{\mathfrak{s}}_{h_0}(\mathbf{x}).$$

Summarizing the above observations we arrive at the following.

Lemma 3.1. *With the above notation fixed*

$$d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d+H^{\mathfrak{s}}_{h_0}=f^{\mathfrak{s}}_1\circ H^{\mathfrak{s}}-H^{\mathfrak{s}}_{h_1}\circ\mathfrak{f}^{\mathfrak{s}}_\infty. \tag{12}$$

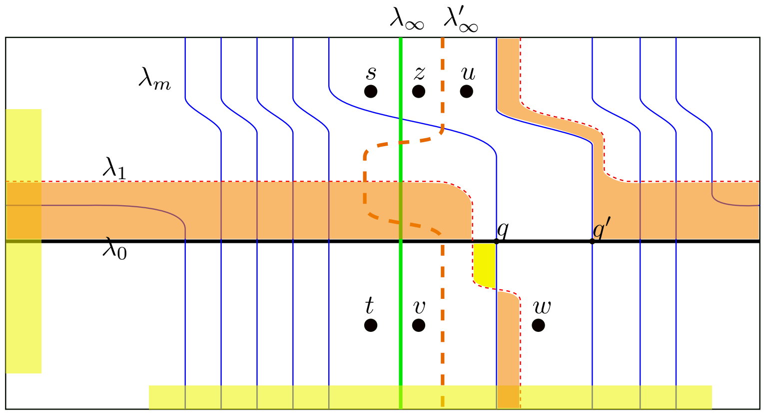

FIGURE 1. The arrangement of the curves on the Heegaard surface. Other curves and handles appear on the shaded yellow area.

Next, we analyse the map $H^{\mathfrak{s}}$ via degenerations of holomorphic squares. For a square class $\square \in \pi_2^0(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \mathbf{y}; u, v, w)$ the moduli space $\mathcal{M}(\square)$ is one dimensional, and has 6 types of boundary ends, corresponding to the degenerations of the domain. Since $\Theta_{0,1}$ and Θ_{g_1} are closed, the 4 types of degenerations of the square class to a square and a bi-gon correspond to the coefficient of $\mathbf{y}$ in $(d \circ H^{\mathfrak{s}} + H^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining boundary ends correspond to a degeneration of $\square$ to a pair of triangle classes. The degenerations $\square = \Delta' \star \Delta$ with $\Delta \in \pi_2(\mathbf{x}, \Theta_{0,1}, \mathbf{z})$ and $\Delta' \in \pi_2(\mathbf{z}, \Theta_{g_1}, \mathbf{y})$ correspond to the coefficient of $\mathbf{y}$ in $(g_1^{\mathfrak{s}} \circ \mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x})$.

The more tricky and interesting part is the contribution of the boundary ends which correspond to the degenerations of the form $\square = \Delta' \star \Delta$ with

$$\Delta' \in \pi_2^0(\Theta_{0,1}, \Theta_{g_1}, \Theta; u, v, w) \quad \text{and} \quad \Delta \in \pi_2^0(\mathbf{x}, \Theta, \mathbf{y}; u, v, w).$$

There are precisely two generators Θ such that there is a corresponding $\Delta' = \Delta_\Theta$ associated with them such that $\mathcal{M}(\Delta_\Theta)$ is non-empty. One of these classes corresponds to $\Theta = \Theta_{g_0}$ and the other one corresponds to the generator Θ'_{g_0} which is obtained from Θ_{g_0} by changing q to the intersection point $q' \in \lambda_0 \cap \lambda_m$ which is next to q (see Figure 1). The total contribution of such boundary ends is thus

$$\sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m} \\ \Delta \in \pi_2^0(\mathbf{x}, \Theta_{g_0}, \mathbf{y}; u, v, w)}} \#\left(\mathcal{M}(\Delta)\right).\mathbf{y} + \sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m} \\ \Delta \in \pi_2^0(\mathbf{x}, \Theta'_{g_0}, \mathbf{y}; u, v, w)}} \#\left(\mathcal{M}(\Delta)\right).\mathbf{y}.$$

Let $g_0^{\mathfrak{s}}(\mathbf{x}) = (g_{0,1}^{\mathfrak{s}}(\mathbf{x}), g_{0,2}^{\mathfrak{s}}(\mathbf{x}))$ denotes the decomposition of $g_0^{\mathfrak{s}}$ in

$$\widehat{\mathrm{CFK}}(K_m, \mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_m, \mathfrak{s} + m).$$

Then the above sum is equal to $g_{0,1}^{\mathfrak{s}}(\mathbf{x}) + G^{\mathfrak{s}}(\mathbf{x})$, where

$$G^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_0, \mathfrak{s}) \longrightarrow \widehat{\mathrm{CFK}}(K_m, \mathfrak{s} + m - 1)$$

is defined by the second sum above. In Section 4 we show that for sufficiently large m and an appropriate Heegaard 5-tuple we may assume that there is an embedding

$$J^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_m, \mathfrak{s} + m) \longrightarrow \widehat{\mathrm{CFK}}(K_m, \mathfrak{s} + m - 1)$$

such that $G^{\mathfrak{s}}(\mathbf{x}) = J^{\mathfrak{s}}(g^{\mathfrak{s}}_{0,2}(\mathbf{x}))$. Set

$$\begin{aligned} G^{\mathfrak{s}}_\infty : \bigoplus_{\mathfrak{t}\in\{\mathfrak{s},\mathfrak{s}+m\}} \widehat{\mathrm{CFK}}(K_m,\mathfrak{t}) &\longrightarrow \bigoplus_{\mathfrak{t}\in\{\mathfrak{s},\mathfrak{s}+m-1\}} \widehat{\mathrm{CFK}}(K_m,\mathfrak{t}),\\ G^{\mathfrak{s}}_\infty(\mathbf{x}_1,\mathbf{x}_2) &:= (\mathbf{x}_1, J^{\mathfrak{s}}(\mathbf{x}_2)). \end{aligned}$$

The above observations imply the following.

Lemma 3.2. *With the above notation fixed we have*

$$d\circ H^{\mathfrak{s}} + H^{\mathfrak{s}}\circ d = g^{\mathfrak{s}}_1\circ \mathfrak{f}^{\mathfrak{s}}_\infty - G^{\mathfrak{s}}_\infty\circ g^{\mathfrak{s}}_0. \tag{13}$$

Let us now consider the Heegaard 5-tuple $\mathcal{H} = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1,\boldsymbol{\beta}_m,\boldsymbol{\beta},\boldsymbol{\beta}';u,v,z)$ where the curves in $\boldsymbol{\beta}' = \{\beta'_1,...,\beta'_g\}$ are small Hamiltonian isotopes of the corresponding curves in $\boldsymbol{\beta}$. Moreover, we assume that the intersection pattern between $\lambda_1,\lambda_m,\lambda_\infty = \beta_g$ and $\lambda'_\infty = \beta'_g$ and the location of u, v and z follows the pattern illustrated in Figure 1. Using the punctured Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u,v,z)$ we may form the chain complex associated with K and $\mathfrak{s}\in \underline{\mathrm{Spin}}^c(Y,K)$. We will thus denote this latter chain complex by $\widehat{\mathrm{CFK}}(K,\mathfrak{s})$. Associated with the Heegaard diagram $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}';u,v,z)$ there is a *top generator* which may be denoted by Θ'_∞. Unlike most of such situations Θ'_∞ is not closed and $d(\Theta'_\infty) = \Theta_\infty$ is the generator which is obtained from Θ'_∞ by changing the choice of intersection point in $\lambda_\infty\cap\lambda'_\infty$. By construction, Θ_∞ is closed. The diagram $\mathcal{H}$ defines a pentagon map

$$\begin{aligned} Q^{\mathfrak{s}} &: \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) \longrightarrow \widehat{\mathrm{CFK}}(K,\mathfrak{s}),\\ Q^{\mathfrak{s}}(\mathbf{x}) &= \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\ \mathfrak{s}(\mathbf{y})=\mathfrak{s}\\ \pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)}} \#\left(\mathcal{M}(\pentagon)\right).\mathbf{y}. \end{aligned}$$

For $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)$ consider the ends of the 1-dimensional moduli space $\mathcal{M}(\pentagon)$, which correspond to the degenerations of the pentagon either to a bi-gon and a pentagon, or to a triangle and a square. There are five types of degenerations of each one of these two forms. Since $\Theta_{g_1},\Theta_{f_1}$ and Θ_∞ are closed, the first five types of degenerations correspond to the coefficient of $\mathbf{y}$ in $(d\circ Q^{\mathfrak{s}} + Q^{\mathfrak{s}}\circ d)(\mathbf{x})$. Other degeneration are of the form $\pentagon = \square\star\Delta$ of one of the following 5 types:

(1) $\square\in\pi_2(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y})$ and $\Delta\in\pi_2(\mathbf{x},\Theta_{g_1},\mathbf{z})$,
(2) $\square\in\pi_2(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\mathbf{z})$ and $\Delta\in\pi_2(\mathbf{z},\Theta_\infty,\mathbf{y})$,
(3) $\square\in\pi_2(\mathbf{x},\Theta_{g_1},\Theta,\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{f_1},\Theta_\infty,\Theta)$,
(4) $\square\in\pi_2(\mathbf{x},\Theta,\Theta_\infty,\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{g_1},\Theta_{f_1},\Theta)$,
(5) $\square\in\pi_2(\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\Theta)$ and $\Delta\in\pi_2(\mathbf{x},\Theta,\mathbf{y})$.

The first type in the above list corresponds to the coefficient of $\mathbf{y}$ in the expression $(I^{\mathfrak{s}}\circ g^{\mathfrak{s}}_1)(\mathbf{x})$, where $I^{\mathfrak{s}}$ is defined by

$$I^{\mathfrak{s}}(\mathbf{z}) = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\ \mathfrak{s}(\mathbf{y})=\mathfrak{s}\\ \square\in\pi_2^{-1}(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)}} \#\left(\mathcal{M}(\square)\right).\mathbf{y}.$$

The second type in the above list corresponds to the coefficient of $\mathbf{y}$ in $(X^{\mathfrak{s}}\circ H^{\mathfrak{s}}_{h_1})(\mathbf{x})$, where $X^{\mathfrak{s}}$ is defined by

$$X^{\mathfrak{s}}(\mathbf{z}) = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\ \mathfrak{s}(\mathbf{y})=\mathfrak{s}\\ \Delta\in\pi_2^{0}(\mathbf{z},\Theta_\infty,\mathbf{y};u,v,z)}} \#\left(\mathcal{M}(\Delta)\right).\mathbf{y}.$$

Considering the local multiplicities around $\lambda_\infty\cap\lambda'_\infty$ one may conclude that there are no triangle classes $\Delta\in\pi_2^0(\mathbf{z},\Theta_\infty,\mathbf{y};u,v,z)$ with positive domain. In particular, $X^{\mathfrak{s}}$ is trivial. There are no triangle classes which contribute in the degenerations of the type (3). The triangles which contribute in degenerations of type (4) come in canceling pairs. Thus the total number of boundary ends corresponding to degenerations of types (3) and (4) is zero. There is a unique square class

$$\square\in\pi_2^{-1}(\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\Theta;u,v,z)$$

with non-trivial contribution to the degenerations of type (5). For this square class $\Theta\in\mathbb{T}_{\beta_1}\cap\mathbb{T}_{\beta'}$ is the top generator, and the class $\square$ has a unique holomorphic representative. The top generator Θ may be used to define

$$\mathfrak{f}_0^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_1,\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K,\mathfrak{s}).$$

The contribution of the degenerations of type 5 thus corresponds to the coefficient of $\mathbf{y}$ in $\mathfrak{f}_0^{\mathfrak{s}}(\mathbf{x})$. The map on homology induced by $\mathfrak{f}_0^{\mathfrak{s}}$ coincides with the map used in the splicing formula of [Ef4].

Define the maps $F_0^{\mathfrak{s}}:M(f_1^{\mathfrak{s}})\to\widehat{\mathrm{CFK}}(K,\mathfrak{s})$ and $F_\infty^{\mathfrak{s}}:M(f_0^{\mathfrak{s}})\to M(f_1^{\mathfrak{s}})$ by

$$F_0^{\mathfrak{s}}(\mathbf{x}_1,\mathbf{x}_2):=I^{\mathfrak{s}}(\mathbf{x}_1),\qquad\text{where }\begin{cases}\mathbf{x}_1\in\widehat{\mathrm{CFK}}(K_m,\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)\\ \mathbf{x}_2\in\widehat{\mathrm{CF}}(Y).\end{cases}$$

$$F_\infty^{\mathfrak{s}}(\mathbf{x}_1,\mathbf{x}_2):=(G_\infty^{\mathfrak{s}}(\mathbf{x}_1),-\mathbf{x}_2),\quad\text{where }\begin{cases}\mathbf{x}_1\in\widehat{\mathrm{CFK}}(K_m,\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m)\\ \mathbf{x}_2\in\widehat{\mathrm{CF}}(Y)\end{cases}$$

With this notation fixed, the outcome of the above observations, together with Lemma 3.1 and Lemma 3.2 is the following theorem.

Theorem 3.3. *With the above notation fixed, the following diagram is commutative, upto chain homotopy*

$$\begin{tikzcd}\widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) \arrow[r, "\mathfrak{f}_\infty^{\mathfrak{s}}"] \arrow[d, "\imath_0^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) \arrow[r, "\mathfrak{f}_0^{\mathfrak{s}}"] \arrow[d, "\imath_1^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \arrow[d, "Id"'] \\ M(f_0^{\mathfrak{s}}) \arrow[r, "F_\infty^{\mathfrak{s}}"] & M(f_1^{\mathfrak{s}}) \arrow[r, "F_0^{\mathfrak{s}}"] & \widehat{\mathrm{CFK}}(K,\mathfrak{s})\end{tikzcd}\quad. \tag{14}$$

Proof. By the discussion preceding the theorem, we have

$$\mathfrak{f}_0^{\mathfrak{s}}-F_0^{\mathfrak{s}}\circ\imath_1^{\mathfrak{s}}=d\circ Q^{\mathfrak{s}}+Q^{\mathfrak{s}}\circ d.$$

This proves the commutativity of the right-hand-side square upto chain homotopy. To prove the commutativity of the left-hand-side square, define

$$R^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\longrightarrow M(f_1^{\mathfrak{s}}),\quad R^{\mathfrak{s}}(\mathbf{x}):=(-H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x})).$$

We thus find

$$\begin{aligned}(d\circ R^{\mathfrak{s}}+R^{\mathfrak{s}}\circ d)(\mathbf{x})&=d(-H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x}))+(R^{\mathfrak{s}}\circ d)(\mathbf{x})\\ &=\big((G_\infty^{\mathfrak{s}}\circ g_0^{\mathfrak{s}}-g_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x}),(d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d-f_1^{\mathfrak{s}}\circ H^{\mathfrak{s}})(\mathbf{x})\big)\\ &=-\big((g_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}}-G_\infty^{\mathfrak{s}}\circ g_0^{\mathfrak{s}})(\mathbf{x}),(H_{h_1}^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}}+H_{h_0}^{\mathfrak{s}})(\mathbf{x})\big)\\ &=(F_\infty^{\mathfrak{s}}\circ\imath_0^{\mathfrak{s}}-\imath_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x}).\end{aligned}$$

The second equality follows from Lemma 3.2, while the third equality follows from Lemma 3.1. This observation completes the proof of Theorem 3.3. $\square$

4. Surgery and splicing formulas for knots

4.1. **Surgery formulas.** Theorem 2.1 implies that $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$ is quasi-isomorphic, for m sufficiently large, to the mapping cone of the chain map

$$f_n^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)\longrightarrow\widehat{\mathrm{CF}}(Y).$$

When the curve λ_{n+m} is very close to the juxtaposition of λ and $(n+m)\beta_g$, and it cuts β_g almost in the middle of the winding region, this mapping cone has a particularly easy description, which is described below.

With the above choice we may assume that associated with every generator $\mathbf{x}$ for the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$, which in turn is a generator of $\widehat{\mathrm{CF}}(Y)$, we obtain $n+m$ generators for $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v)$. These $n+m$ generators will be denoted by $\mathbf{x}_{1-l},\mathbf{x}_{2-l},...,\mathbf{x}_{m+n-l}$, where $l=\lfloor m/2\rfloor$ and $\mathbf{x}_i$ is on the left of β_g if $i<0$ and is on the right of β_g otherwise. The rest of generators for the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v)$ are in correspondence with the generators $\mathbf{y}$ of $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v)$. Every such generator will be denote by $\widehat{\mathbf{y}}$. With this notation fixed we have

$$\mathfrak{s}(\mathbf{x}_i)=\begin{cases}\mathfrak{s}(\mathbf{x})+i & \text{if } i\geq 0\\ \mathfrak{s}(\mathbf{x})+n+m+i & \text{if } i<0\end{cases}\quad\text{and}\quad \mathfrak{s}(\widehat{\mathbf{y}})=\mathfrak{s}(\mathbf{y})+n+\left\lceil\frac{m}{2}\right\rceil.$$

Restricting our attention to the relative Spin^c classes $\mathfrak{s}$ and $\mathfrak{s}+m$ we find

$$\begin{aligned}\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})&=\left\langle\mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})}\;\middle|\;\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})\leq\mathfrak{s}\right\rangle,\\ \widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)&=\left\langle\mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})-n}\;\middle|\;\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\right\rangle,\end{aligned}$$

If the curve λ_{n+m} is sufficiently close to the juxtaposition $\lambda\star(m+n)\beta_g$ the first complex is identified with the sub-complex

$$\left\langle\mathbf{x}\;\middle|\;\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})\leq\mathfrak{s}\right\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$, while the restriction of the map $f^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})$ is identified with the inclusion of the aforementioned sub-complex in $\widehat{\mathrm{CF}}(Y)$. Similarly, the second complex is identified with the sub-complex

$$\left\langle\mathbf{x}\;\middle|\;\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\right\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$ while the restriction of the map $\overline{f}^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)$ is identified with the inclusion of the aforementioned sub-complex in $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$.

Let $C=C_K$ denote the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex generated by triples $[\mathbf{x},i,j]$ with $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\ i,j\in\mathbb{Z}$ and $\mathfrak{s}(\mathbf{x})-i+j=0$. The differential of C is defined by

$$\begin{aligned}d[\mathbf{x},i,j]&=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\phi\in\pi_2^1(\mathbf{x},\mathbf{y})}\#\left(\widehat{\mathcal{M}}(\phi)\right)[\mathbf{y},i-n_u(\phi),j-n_s(\phi)]\\ &=\sum_{a,b=0}^{\infty}[d^{a,b}(\mathbf{x}),i-a,j-b].\end{aligned}$$

Since $d\circ d=0$ we conclude that $d^{0,0}\circ d^{0,0}=0$, while

$$\begin{aligned}
&d^{0,1}\circ d^{0,0}+d^{0,0}\circ d^{0,1}=0,\\
&d^{1,0}\circ d^{0,0}+d^{0,0}\circ d^{1,0}=0 \quad \text{and}\\
&d^{1,1}\circ d^{0,0}+d^{0,0}\circ d^{1,1}+d^{0,1}\circ d^{1,0}+d^{1,0}\circ d^{0,1}=0.
\end{aligned}\tag{15}$$

Following [OS3] (or the notation set in the introduction) $\widehat{\mathrm{CF}}(Y)$ is identified as $C\{j=0\}$, while $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})$ and $\widehat{\mathrm{CF}}(K_{n+m},\mathfrak{s}+m)$ are identified with

$$C\{i\le\mathfrak{s},j=0\}\ \text{ and }\ C\{i=0,j\le n-\mathfrak{s}-1\},$$

respectively. There is a chain homotopy equivalence Ξ from $C\{i=0\}$ to $C\{j=0\}$. The following is thus just a re-statement of Theorem 2.1.

Theorem 4.1. *Fix the above notation and a class $\mathfrak{s}\in\mathbb{Z}=\underline{\mathrm{Spin}}^c(Y,K)$. Then $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$ is quasi-isomorphic to the mapping cone $M(i_n^{\mathfrak{s}})$ of*

$$\begin{aligned}
&i_n^{\mathfrak{s}}:C\{i\le\mathfrak{s},j=0\}\oplus C\{i=0,j\le n-\mathfrak{s}-1\}\longrightarrow C\{j=0\},\\
&i_n^{\mathfrak{s}}([\mathbf{x},i,0],[\mathbf{y},0,j]):=[\mathbf{x},i,0]+\Xi[\mathbf{y},0,j].
\end{aligned}$$

4.2. **The bypass homomorphisms.** We now turn to understanding the maps $F_0^{\mathfrak{s}}$ and $F_\infty^{\mathfrak{s}}$ (which will be called the *bypass homomorphisms*) under the above identifications. To understand $F_0^{\mathfrak{s}}$, one should identify $I^{\mathfrak{s}}$ on

$$\widehat{\mathrm{CFK}}(K_m,\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)=C\{i\le\mathfrak{s},j=0\}\oplus C\{i=0,j\le-\mathfrak{s}\}.$$

Let $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $\mathbf{x}_i$ be the corresponding generator in $\widehat{\mathrm{CFK}}(K_m)$ and suppose that $\square\in\pi_2^{-1}(\mathbf{x}_i,\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)$ contributes to $I^{\mathfrak{s}}$. Looking at local coefficients implies that $i=-1$. In particular, $\mathfrak{s}(\mathbf{x})=\mathfrak{s}(\mathbf{y})=\mathfrak{s}$ and $\mathbf{x}_{-1}$ corresponds to the generator $[\mathbf{x},0,-\mathfrak{s}]\in C\{i=0,j\le-\mathfrak{s}\}$. There is a particular square class with very small domain which connects $\mathbf{x}_{-1},\Theta_{f_1},\Theta_\infty$ and $\mathbf{x}$ and has non-trivial contribution to $I^{\mathfrak{s}}$. Considering the energy filtration and modifying $\widehat{\mathrm{CFK}}(K,\mathfrak{s})=C\{i=0,j=-\mathfrak{s}\}$ by the chain map $I^{\mathfrak{s}}|_{C\{i=0,j=-\mathfrak{s}\}}$ which is a change of basis , we may thus assume that $F_0^{\mathfrak{s}}$ is induced by projecting the factor $C\{i=0,j\le-\mathfrak{s}\}$ in the mapping cone of $\imath_1^{\mathfrak{s}}$ over the quotient complex $C\{i=0,j=-\mathfrak{s}\}=\widehat{\mathrm{CFK}}(K,\mathfrak{s})$.

In order to study the map $F_\infty^{\mathfrak{s}}$ we need to understand the map

$$G^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1).$$

Local considerations imply that for a triangle class $\Delta\in\pi_2^0(\mathbf{x},\Theta'_{g_0},\mathbf{y};u,v,w)$ which has non-trivial contribution to $G^{\mathfrak{s}}$ we have $\mathbf{y}=\mathbf{z}_i$ with $\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $i\le-2$. Every such Δ corresponds to a triangle class $\Delta'\in\pi_2^0(\mathbf{x},\Theta_{g_0},\mathbf{z}_{i+1};u,v,w)$, and if λ_m is sufficiently close to the juxtaposition $\lambda\star m\beta_g$ and m is sufficiently large the moduli spaces $\mathcal{M}(\Delta)$ and $\mathcal{M}(\Delta')$ may in fact be identified. Note that $\mathfrak{s}(\mathbf{z}_{i+1})=\mathfrak{s}(\mathbf{z}_i)+1=\mathfrak{s}+m$ and that these latter disk classes Δ' are the disk classes which contribute to the holomorphic triangle map $g_{0,2}^{\mathfrak{s}}$. The image of $G^{\mathfrak{s}}$ is thus in

$$C\{i=0,j\le-\mathfrak{s}-1\}\subset\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)=C\{i=0,j\le-\mathfrak{s}\},$$

and if

$$\begin{aligned}
J^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m)&=C\{i=0,j\le-\mathfrak{s}-1\}\\
&\longrightarrow\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)=C\{i=0,j\le-\mathfrak{s}\}
\end{aligned}$$

denotes the inclusion, $G^{\mathfrak{s}}(\mathbf{x}) = J^{\mathfrak{s}}(g^{\mathfrak{s}}_{0,2}(\mathbf{x}))$. This implies the following theorem.

Theorem 4.2. *Under the identification of $\widehat{\mathrm{CFK}}(K_\bullet,\mathfrak{s})$ with $M(i^{\mathfrak{s}}_\bullet)$ for $\bullet = 0, 1$, $F^{\mathfrak{s}}_\infty$ is given by the inclusion of $M(i^{\mathfrak{s}}_0)$ in $M(i^{\mathfrak{s}}_1)$ as a sub-complex, while $F^{\mathfrak{s}}_0$ is given by the quotient map. In particular, we have a short exact sequence*

$$0 \longrightarrow M(i^{\mathfrak{s}}_0) \xrightarrow{F^{\mathfrak{s}}_\infty = \hookrightarrow} M(i^{\mathfrak{s}}_1) \xrightarrow{F^{\mathfrak{s}}_0} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) = \frac{M(i^{\mathfrak{s}}_1)}{M(i^{\mathfrak{s}}_0)} \longrightarrow 0.$$

Theorem 4.2 implies that the second row in (14) is part of a short exact sequence. The discussion preceding Theorem 4.6 in [Ef4] implies that the initial Heegaard diagram may be chosen so that the first row is also completed to a short exact sequence. We thus have the following commutative diagram (upto chain homotopy):

$$(16)\qquad \begin{tikzcd} 0 \arrow[r] & \widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) \arrow[r, "\mathfrak{f}^{\mathfrak{s}}_\infty"] \arrow[d, "i^{\mathfrak{s}}_0"'] & \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) \arrow[r, "\mathfrak{f}^{\mathfrak{s}}_0"] \arrow[d, "i^{\mathfrak{s}}_1"'] & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \arrow[r] \arrow[d, "Id"'] & 0 \\ 0 \arrow[r] & M(i^{\mathfrak{s}}_0) \arrow[r, "F^{\mathfrak{s}}_\infty"] & M(i^{\mathfrak{s}}_1) \arrow[r, "F^{\mathfrak{s}}_0"] & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \arrow[r] & 0 \end{tikzcd} .$$

In particular, in the level of homology, the connecting homomorphism of the short exact sequence in the second row of (16) is identified with the connecting homomorphism $\mathfrak{f}^{\mathfrak{s}}_1$ of the first row, which is used in the splicing formula of [Ef4]. A completely similar argument identifies $\bar{\mathfrak{f}}^{\mathfrak{s}}_\infty$ with the inclusion map $\overline{F}^{\mathfrak{s}}_\infty$ from $M(i^{\mathfrak{s}-1}_0)$ to $M(i^{\mathfrak{s}}_1)$ and $\bar{\mathfrak{f}}^{\mathfrak{s}}_0$ with the quotient map $\overline{F}^{\mathfrak{s}}_0$ to $\widehat{\mathrm{CFK}}(K,\mathfrak{s})$, while $\bar{\mathfrak{f}}^{\mathfrak{s}}_1$ is identified with the connecting homomorphism of the short exact sequence

$$(17)\qquad 0 \longrightarrow M(i^{\mathfrak{s}-1}_0) \xrightarrow{\overline{F}^{\mathfrak{s}}_\infty} M(i^{\mathfrak{s}}_1) \xrightarrow{\overline{F}^{\mathfrak{s}}_0} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0.$$

Proof. (of Theorem 1.5) For $\bullet = 0, 1$, let us define

$$C_\bullet(K) = \bigoplus_{\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)} C_\bullet(K,\mathfrak{s}), \quad \text{where } C_\bullet(K,\mathfrak{s}) = M(i^{\mathfrak{s}}_\bullet).$$

Let $C_\infty(K) = \bigoplus_{\mathfrak{s}} C_\infty(K,\mathfrak{s})$, where $C_\infty(K,\mathfrak{s}) = C\{i = \mathfrak{s}, j = 0\}$. The chain maps $F_\infty, \overline{F}_\infty\colon C_0(K) \to C_1(K)$ and $F_0, \overline{F}_0\colon C_1(K) \to C_\infty(K)$ sit in the short exact sequences

$$0 \longrightarrow C_0(K) \xrightarrow{F_\infty} C_1(K) \xrightarrow{F_0} C_\infty(K) \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow C_0(K) \xrightarrow{\overline{F}_\infty} C_1(K) \xrightarrow{\overline{F}_0} C_\infty(K) \longrightarrow 0$$

The maps induced by $F_\bullet$ and $\overline{F}_\bullet$ are $\mathfrak{f}_\bullet$ and $\bar{\mathfrak{f}}_\bullet$, respectively. Thus, Proposition 7.2 from [Ef4] may be applied here to complete the proof of Theorem 1.5. □

5. The linear algebra of bypass homomorphisms

5.1. **Nilpotent compositions.** Let K be a knot inside the homology sphere Y and let us continue to use the notation of the previous sections.

Lemma 5.1. *Let* $\mathbf{x} \in \widehat{\mathrm{CFK}}(K,\mathfrak{s})$ *be a closed element and* $[\mathbf{x}]$ *denote the class represented by* $\mathbf{x}$ *in* $\widehat{\mathrm{HFK}}(K,\mathfrak{s})$*. Then*

$$(\overline{F}_0^{\mathfrak{s}-1} \circ F_\infty^{\mathfrak{s}-1} \circ \overline{F}_1^{\mathfrak{s}})[\mathbf{x}] = [d^{1,0}(\mathbf{x})] \quad \textit{and} \quad (F_0^{\mathfrak{s}+1} \circ \overline{F}_\infty^{\mathfrak{s}+1} \circ F_1^{\mathfrak{s}})[\mathbf{x}] = [d^{0,1}(\mathbf{x})].$$

Proof. Since $\overline{F}_1^{\mathfrak{s}}$ is the connecting homomorphism associated with the short exact sequence (17), in order to compute $\overline{F}_1^{\mathfrak{s}}[\mathbf{x}]$ note that $\mathbf{x}$ is the image of $([\mathbf{x},\mathfrak{s},0],0,0) \in M(i_1^{\mathfrak{s}})$ under the quotient map. The differential of $M(i_1^{\mathfrak{s}})$ takes this element to

$$\left(\sum_{i=0}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right) \in M(i_1^{\mathfrak{s}}).$$

Since $d^{0,0}(\mathbf{x})=0$ this latter element is in $M(i_0^{\mathfrak{s}-1})$. $F_\infty^{\mathfrak{s}-1}$ is the inclusion, thus

$$\left(F_\infty^{\mathfrak{s}-1} \circ \overline{F}_1^{\mathfrak{s}}\right)[\mathbf{x}] = \left(\sum_{i=1}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right) \in M(i_1^{\mathfrak{s}-1})$$

The projection map $\overline{F}_0^{\mathfrak{s}-1}$ takes this latter element to the closed element $d^{1,0}(\mathbf{x})$ in $\widehat{\mathrm{CFK}}(K,\mathfrak{s}-1)$. The second claim is proved similarly. ☐

Corollary 5.2. *For every relative* Spin^c *class* $\mathfrak{s}$ *the map*

$$\bar{\mathfrak{f}}_0 \circ \mathfrak{f}_\infty \circ \bar{\mathfrak{f}}_1 \circ \mathfrak{f}_0 \circ \bar{\mathfrak{f}}_\infty \circ \mathfrak{f}_1|_{\widehat{\mathrm{HFK}}(K,\mathfrak{s})} : \widehat{\mathrm{HFK}}(K,\mathfrak{s}) \longrightarrow \widehat{\mathrm{HFK}}(K,\mathfrak{s})$$

is nilpotent.

Proof. It suffices to show that $F = \overline{F}_0 \circ F_\infty \circ \overline{F}_1 \circ F_0 \circ \overline{F}_\infty \circ F_1$ is nilpotent. However, by Lemma 5.1, for $\mathbf{x} \in \widehat{\mathrm{HFK}}(K,\mathfrak{s})$ we have

$$\begin{aligned} & F[\mathbf{x}] = [d^{1,0}(d^{0,1}(\mathbf{x}))] \\ \Rightarrow \quad & F^n[\mathbf{x}] = \left[\left(d^{1,0}\circ d^{0,1}\right)^n(\mathbf{x})\right] = \left[\left(\left(d^{1,0}\right)^n \circ \left(d^{0,1}\right)^n\right)(\mathbf{x})\right], \end{aligned}$$

where the last equality follows by an inductive use of (15). Since $[(d^{0,1})^n(\mathbf{x})]$ is in $\widehat{\mathrm{HFK}}(K,\mathfrak{s}+n)$ and for large values of n $\widehat{\mathrm{HFK}}(K,\mathfrak{s}+n)$ is trivial, it follows that F^n is trivial if n is sufficiently large (e.g. if $n > 2g$ where g is the genus of K). ☐

5.2. **Block decomposition for bypass homomorphisms.** Let us assume that the chain complex C is defined from the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v)$. Changing the role of the two punctures gives the duality maps

$$\tau_\bullet = \tau_\bullet(K) : \mathbb{H}_\bullet(K) \longrightarrow \mathbb{H}_\bullet(K), \qquad \bullet \in \{0,1,\infty\}.$$

These duality maps take $\mathbb{H}_\bullet(K,\mathfrak{s})$ to $\mathbb{H}_\bullet(K,-\mathfrak{s})$ if $\bullet = 1,\infty$, and to $\mathbb{H}_0(K,1-\mathfrak{s})$ when $\bullet = 0$. Furthermore, we have $\tau_\bullet \circ \tau_\bullet = Id$. Following the notation of [Ef4], in a basis for $\mathbb{H}_\bullet(K)$ where $\mathfrak{f}_\bullet$ takes the block form $\left(\begin{smallmatrix}0 & 0\\ I & 0\end{smallmatrix}\right)$, we assume that

$$\tau_\bullet = \begin{pmatrix} A_\bullet & B_\bullet \\ C_\bullet & D_\bullet \end{pmatrix} \quad \bullet \in \{0,1,\infty\}. \tag{18}$$

As observed in [Ef4], one can then compute

$$\bar{\mathfrak{f}}_0 = \tau_\infty \circ \mathfrak{f}_0 \circ \tau_1, \quad \bar{\mathfrak{f}}_1 = \tau_0 \circ \mathfrak{f}_1 \circ \tau_\infty \quad \text{and} \quad \bar{\mathfrak{f}}_\infty = \tau_1 \circ \mathfrak{f}_\infty \circ \tau_0.$$

Define $X_\bullet = X_\bullet(K)$ by $X_0 = B_1B_0B_\infty, X_1 = B_\infty B_1 B_0$ and $X_\infty = B_0B_\infty B_1$. We denote the rank of $F_\bullet$ by $a_\bullet = a_\bullet(K)$, for $\bullet = 0, 1, \infty$. Thus a_1, a_∞ and a_0+1 have the same parity. Note that B_0, B_1 and B_∞ are (respectively) matrices of size

$$a_\infty \times a_1,\ \ a_0 \times a_\infty \ \text{ and } \ a_1 \times a_0.$$

Lemma 5.3. *If K is a knot of genus $g > 0$ then $B_\bullet \neq 0$ for $\bullet \in \{0, 1, \infty\}$. In particular, $a_\bullet > 0$.*

Proof. Since $H_*(M(i_0^g)) = 0$ by Theorem 4.1, the map

$$F_0^g : \mathbb{H}_1(K, g) \longrightarrow \mathbb{H}_\infty(K, g)$$

is an isomorphism. From here and by duality $\overline{F}_0^{-g}$ is also an isomorphism. Similarly,

$$\begin{aligned} &H_*\left(M\left(i_1^{-g}\right)\right) \simeq \widehat{\mathrm{HFK}}(K, -g) \ \text{ and}\\ &H_*\left(M\left(i_0^{-g}\right)\right) \simeq \widehat{\mathrm{HFK}}(K, -g) \oplus \widehat{\mathrm{HFK}}(K, -g). \end{aligned}$$

Thus F_∞^{-g} is surjective, i.e. F_0^{-g} is trivial, implying that:

- $\mathrm{Ker}(F_0) \setminus \mathrm{Ker}(\overline{F}_0)$ is non-empty.
- $\mathrm{Im}(\overline{F}_0) \setminus \mathrm{Im}(F_0)$ is non-empty.

Returning to the matrix presentations, the first claim above implies that

$$\exists \ \binom{a}{b} \in \mathbb{H}_1(K) \ \ \text{s.t.} \ \begin{cases} 0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\binom{a}{b} \\ \\ 0 \neq \begin{pmatrix} A_\infty & B_\infty \\ C_\infty & D_\infty \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\binom{a}{b} \end{cases}$$

Thus $a = 0$ and $\binom{B_\infty B_1 b}{D_\infty B_1 b} \neq 0$. In particular, $B_1 \neq 0$. Similarly, the second claim above implies that $\mathrm{Ker}(\overline{F}_1) \setminus \mathrm{Ker}(F_1)$ is non-empty and thus $B_\infty \neq 0$.

For non-triviality of B_0, choose $x \in \mathbb{H}_\infty(K, g)$. This element may be represented by $y = [x, g, 0] \in C\{j = 0\}$. Thus, $d^{*,0}y \in C\{i < g, j = 0\}$ and $\overline{F}_1^g(x) = (d^{*,0}y, y, 0) \in M(i_0^{g-1})$ is thus in the kernel of $\overline{F}_\infty$. If $F_\infty^{g-1}(\overline{F}_1^g(x)) = 0$ then $\overline{F}_1^g(x) = F_1^{g-1}(x')$ for some $x' \in \mathbb{H}_\infty(K, g-1)$. In other words, if we denote the dual of $[x', 0, 1-g]$ by $z \in C\{i \le 1-g, j = 0\}$, the above equality implies

$$\exists \ \begin{cases} \overline{y} \in C\{i < g, j = 0\}, \\ \overline{z} \in C\{i < 1-g, j = 0\} \end{cases} \quad \text{s.t} \ \begin{cases} d^{*,0}(y - \overline{y}) = 0 \\ d^{*,0}(z - \overline{z}) = 0 \\ (y - \overline{y}) + (z - \overline{z}) \text{ is exact.} \end{cases}$$

Note that $-\overline{y} + z - \overline{z} \in C\{i < g, j = 0\}$ while y represents a non-trivial element in the homology of the quotient

$$C\{i = g, j = 0\} = \frac{C\{i \le g, j = 0\}}{C\{i < g, j = 0\}}.$$

Thus $(y - \overline{y}) + (z - \overline{z})$ can not be exact, and $\mathrm{Ker}(F_\infty) \setminus \mathrm{Ker}(\overline{F}_\infty)$ can not be trivial. From here, an argument similar to the preceding two cases implies $B_0 \neq 0$. □

Lemma 5.4. *With the above notation fixed, the three matrices $X_\bullet$ are all nilpotent for $\bullet \in \{0, 1, \infty\}$. In particular, if the knot K is non-trivial both the kernel and the cokernel of $X_\bullet$ are non-trivial.*

Proof. The first claim is a direct consequence of Corollary 5.2 once we represent $F_\bullet = F_\bullet(K)$ as $\left(\begin{smallmatrix}0&0\\I&0\end{smallmatrix}\right)$ and note that $\overline{F}_\bullet = F_\bullet(K)$ are given by

$$\overline{F}_0 = \tau_\infty F_0 \tau_1, \quad \overline{F}_1 = \tau_0 F_1 \tau_\infty \quad \text{and} \quad \overline{F}_\infty = \tau_1 F_\infty \tau_0,$$

which implies $\overline{F}_0 F_\infty \overline{F}_1 F_0 \overline{F}_\infty F_1 = \left(\begin{smallmatrix}(X_1)^2&0\\\star&0\end{smallmatrix}\right)$. Thus there is a positive integer N so that $X_1^N = 0$. As a consequence $X_\bullet^{N+1} = 0$ for $\bullet = 0, 1, \infty$. The second claim is a consequence of the first claim unless $a_\bullet = 0$. However, by Lemma 5.3, $a_\bullet > 0$. $\square$

Definition 5.5. *The knot K inside the homology sphere Y is called* full-rank *if all three matrices $B_0(K), B_1(K)$ and $B_\infty(K)$ are full rank.*

If $P_\bullet$ is an invertible $a_\bullet \times a_\bullet$ matrix and the matrices $Y_\bullet$ are arbitrary matrices of correct size, we may choose a change of basis for either of $\mathbb{H}_0(K), \mathbb{H}_1(K)$ and $\mathbb{H}_\infty(K)$ which is given by the invertible matrices

$$\mathbb{P}_0 = \begin{pmatrix} P_\infty & 0 \\ Y_0 & P_1 \end{pmatrix}, \quad \mathbb{P}_1 = \begin{pmatrix} P_0 & 0 \\ Y_1 & P_\infty \end{pmatrix} \text{ and } \mathbb{P}_\infty = \begin{pmatrix} P_1 & 0 \\ Y_\infty & P_0 \end{pmatrix}, \tag{19}$$

respectively. The block forms $F_\bullet = \left(\begin{smallmatrix}0&0\\I&0\end{smallmatrix}\right)$ remain unchanged under such a change of basis. A simultaneous change of basis of the form illustrated in (19) is called an *admissible* change of basis. The following lemma will be useful through our forthcoming discussions.

Lemma 5.6. *Suppose that K is a knot in a homology sphere and for $\bullet \in \{0, 1, \infty\}$ let $\tau_\bullet$ denote $\tau_\bullet(K)$ and $X_\bullet$ denote the matrix $X_\bullet(K)$. Choose*

$$(\circ, \bullet, *) \in \{(0, 1, \infty), (1, \infty, 0), (\infty, 0, 1)\}.$$

(1) *If $B_\circ(K), B_\bullet(K)$ are injective and $B_*(K)$ is surjective, after an admissible change of basis we may assume that*

$$\tau_\circ = \left(\begin{array}{cc|c} 0 & 0 & I \\ 0 & \star & 0 \\ \hline I & 0 & 0 \end{array}\right), \ \tau_\bullet = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & \star & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right) \ \textit{and} \ \tau_* = \left(\begin{array}{c|ccc} 0 & X_\bullet & \star & \star \\ \hline \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{array}\right) \tag{20}$$

(2) *If $B_\circ(K), B_\bullet(K)$ are surjective and $B_*(K)$ is injective, after an admissible change of basis we may assume that*

$$\tau_\bullet = \left(\begin{array}{c|cc} 0 & 0 & I \\ \hline 0 & \star & 0 \\ I & 0 & 0 \end{array}\right), \quad \tau_\circ = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline 0 & 0 & \star & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right) \ \textit{and} \ \tau_* = \left(\begin{array}{ccc|c} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & X_\circ \\ \hline \star & \star & \star & 0 \end{array}\right) \tag{21}$$

Proof. The proof consists of straight-forward linear algebra. $\square$

6. Splicing and homology sphere L-spaces

6.1. **Special pairs.** Given an arbitrary matrix M denote the rank of $\mathrm{Ker}(M)$ by $k(M)$, denote the rank of $\mathrm{Coker}(M)$ by $c(M)$ and set $i(M) = k(M) + c(M)$. The matrices M_1 and M_2 are called *equivalent* if $k(M_1) = k(M_2)$ and $c(M_1) = c(M_2)$. If $M^\star \in M_{n_\star \times m_\star}(\mathbb{F})$ for $\star = 1, 2$ are a pair of matrices, $M^1 \otimes M^2 \in M_{n_1 n_2 \times m_1 m_2}(\mathbb{F})$ is the associated map from $\mathbb{F}^{m_1 m_2} = \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2}$ to $\mathbb{F}^{n_1 n_2} = \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$.

Let $Y = Y(K_1, K_2)$ denote the three-manifold obtained by splicing the complements of $K_1 \subset Y_1$ and $K_2 \subset Y_2$, where Y_1 and Y_2 are homology spheres. For $\square \in \{A, B, C, D, X, \tau\}$, $\bullet \in \{0, 1, \infty\}$ and $\star \in \{1, 2\}$ let $\square_\bullet^\star = \square_\bullet(K_\star)$. Proposition 5.4 from [Ef4] and the discussion following it give the following.

Proposition 6.1. *If K_i is a knot inside the homology sphere Y_i for $i = 1, 2$,*

$$\mathrm{rnk}\ \widehat{\mathrm{HF}}(Y(K_1, K_2); \mathbb{F}) = i(\mathfrak{D}(K_1, K_2)),$$

where the matrix $\mathfrak{D}(K_1, K_2)$ is given by

$$\begin{pmatrix}
D^1_\infty B^1_1 \otimes B^2_1 A^2_0 & B^1_1 A^1_0 \otimes I & B^1_1 B^1_0 \otimes I & D^1_\infty A^1_1 \otimes B^2_1 A^2_0 & I \otimes B^2_1 B^2_0 & 0 \\
I \otimes B^2_\infty B^2_1 & D^1_1 A^1_0 \otimes B^2_\infty A^2_1 & D^1_1 B^1_0 \otimes B^2_\infty A^2_1 & 0 & B^1_0 B^1_\infty \otimes I & B^1_0 A^1_\infty \otimes I \\
I \otimes D^2_\infty B^2_1 & \begin{matrix} I \otimes I + \\ D^1_1 A^1_0 \otimes D^2_\infty A^2_1 \end{matrix} & D^1_1 B^1_0 \otimes D^2_\infty A^2_1 & 0 & 0 & 0 \\
B^1_\infty B^1_1 \otimes I & 0 & I \otimes B^2_0 B^2_\infty & B^1_\infty A^1_1 \otimes I & \begin{matrix} D^1_0 B^1_\infty \otimes B^2_0 A^2_\infty \\ + X^1_1 B^1_\infty \otimes B^2_0 X^2_1 \end{matrix} & \begin{matrix} D^1_0 A^1_\infty \otimes B^2_0 A^2_\infty \\ + X^1_1 A^1_\infty \otimes B^2_0 X^2_1 \end{matrix} \\
D^1_\infty B^1_1 \otimes D^2_1 A^2_0 & 0 & 0 & \begin{matrix} I \otimes I + \\ D^1_\infty A^1_1 \otimes D^2_1 A^2_0 \end{matrix} & I \otimes D^2_1 B^2_0 & 0 \\
0 & 0 & I \otimes D^2_0 B^2_\infty & 0 & \begin{matrix} D^1_0 B^1_\infty \otimes D^2_0 A^2_\infty \\ + X^1_1 B^1_\infty \otimes D^2_0 X^2_1 \end{matrix} & \begin{matrix} I \otimes I + \\ D^1_0 A^1_\infty \otimes D^2_0 A^2_\infty \\ + X^1_1 A^1_\infty \otimes D^2_0 X^2_1 \end{matrix}
\end{pmatrix},$$

Definition 6.2. *The pair (K_1, K_2) is called a* special pair *if*

$$\widehat{\mathrm{HF}}(Y(K_1, K_2); \mathbb{F}) = \mathbb{F}.$$

Let us assume, throughout this section, that (K_1, K_2) is a special pair, which is the case if and only if $i(\mathfrak{D}(K_1, K_2)) = 1$. Let $k_\star^\bullet = k(B_\star^\bullet)$ and $c_\star^\bullet = c(B_\star^\bullet)$, for $\star \in \{0, 1, \infty\}$ and $\bullet = 1, 2$. Define $\imath : \{0, 1, \infty\} \to \{0, 1, \infty\}$ by $\imath(0) = \infty, \imath(1) = 1$ and $\imath(\infty) = 0$. Let $\mathfrak{D} = \mathfrak{D}(K_1, K_2)$ and note that the cokernel of $\mathfrak{D}$ includes a subspace $\mathrm{C}(\mathfrak{D})$ and its kernel includes a subspace $\mathrm{K}(\mathfrak{D})$ which are isomorphic to

$$\bigoplus_{\bullet \in \{0,1,\infty\}} \mathrm{Coker}(B^1_\bullet) \otimes \mathrm{Coker}(B^2_{\imath(\bullet)}) \quad \text{and} \quad \bigoplus_{\bullet \in \{0,1,\infty\}} \mathrm{Ker}(B^1_\bullet) \otimes \mathrm{Ker}(B^2_{\imath(\bullet)})$$

respectively, and correspond to the first, second and fourth rows, and to the first, third and fifth columns, respectively. Moreover, if $A^1_\infty \otimes D^2_0 + D^1_0 \otimes A^2_\infty = 0$ (which may be assumed after an admissible change of basis if $c^1_\infty k^2_0 = k^1_0 c^2_\infty = 0$) the cokernel also includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$ and the kernel includes a subspace isomorphic to $\mathrm{Ker}(B^1_0) \otimes \mathrm{Ker}(B^2_0)$. Denote the ranks of $\mathrm{K}(\mathfrak{D})$ and $\mathrm{C}(\mathfrak{D})$ by $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$, respectively. Thus $k(\mathfrak{D}) + c(\mathfrak{D}) \leq 1$ and

$$\widehat{k}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} k^1_\bullet k^2_{\imath(\bullet)} \leq k(\mathfrak{D}) \quad \text{and} \quad \widehat{c}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} c^1_\bullet c^2_{\imath(\bullet)} \leq c(\mathfrak{D}).$$

Proposition 6.3. *If (K_1, K_2) is a special pair, then possibly after interchanging K_1 and K_2, one of the following is the case:*

- (G) *K_1 is full-rank.*
- (S-1) *The matrix B^2_0 is invertible, B^1_0 is surjective and B^1_1 and B^2_∞ are injective.*
- (S-2) *The matrix B^2_0 is invertible, B^1_0 is injective and B^1_1 and B^2_∞ are surjective.*

Proof. We assume that (K_1, K_2) is a special pair, while none of K_1 and K_2 is full-rank. Let us first assume that both $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ are zero. From the above assumption we find $k^1_\bullet k^2_{\imath(\bullet)} = c^1_\bullet c^2_{\imath(\bullet)} = 0$ for $\bullet = 0, 1, \infty$. If $B^1_\bullet$ is not a full rank

matrix then both $c^1_\bullet$ and $k^1_\bullet$ are non-zero. From here $k^2_{\imath(\bullet)}=c^2_{\imath(\bullet)}=0$, i.e. $B^2_{\imath(\bullet)}$ is invertible. Since the parity of a^2_0 is different from the parity of a^2_1 and a^2_∞, the matrices B^2_1 and B^2_∞ can not be square matrices. Thus $\imath(\bullet)=0$ and $\bullet=\infty$. In other words, we conclude that B^1_0 and B^1_1 are full-rank and B^2_0 is invertible, while B^2_∞ is not full-rank. Similarly, we may conclude that B^2_1 is full-rank and B^1_0 is invertible, while B^1_∞ is not full-rank. Moreover, since $c^1_1c^2_1=k^1_1k^2_1=0$, precisely one of B^1_1 and B^2_1 is injective, and the other one is surjective. Without loosing on generality we may thus assume that:

- B^1_0 and B^2_0 are invertible, B^1_1 is injective and B^2_1 is surjective.
- None of B^1_∞ and B^2_∞ is full-rank.

In particular, $k^1_\infty>c^1_\infty>0$ and $c^2_\infty>k^2_\infty>0$. Since B^1_0 and B^2_0 are both invertible we may assume that $D^1_0=0$ and $D^2_0=0$. From here the cokernel of $\mathfrak{D}$ includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty)\otimes\mathrm{Coker}(B^2_\infty)$, which is of size $c^1_\infty c^2_\infty\geq 2$. This implies that (K_1,K_2) is not special.

From this contradiction, we conclude that one of $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ is non-zero. Suppose that $\widehat{c}(\mathfrak{D})=1$ and $\widehat{k}(\mathfrak{D})=0$. For some $\bullet\in\{0,1,\infty\}$ we thus have $c^1_\bullet=c^2_{\imath(\bullet)}=1$ while $k^1_\bullet k^2_{\imath(\bullet)}=0$ and for $\star\neq\bullet$ we have $c^1_\star c^2_{\imath(\star)}=k^1_\star k^2_{\imath(\star)}=0$. Without loosing on generality we may assume that $k^1_\bullet=0$. Thus $B^1_\bullet$ is injective with a 1-dimensional cokernel. In particular, the parity of the number of rows and the number of columns for $B^1_\bullet$ are different, i.e. $\bullet\neq 0$. Thus $c^1_0c^2_\infty=k^1_0k^2_\infty=0$. Since B^2_∞ is not a square matrix, at least one of c^2_∞ and k^2_∞ is non-zero, implying that at least one of c^1_0 and k^1_0 is zero, i.e. B^1_0 is full-rank. The assumption that K_1 is not full-rank implies that $B^1_\star$ is not full-rank, where $\{\star\}=\{1,\infty\}\setminus\{\bullet\}$. From here $c^1_\star,k^1_\star>0$. Together with $c^1_\star c^2_{\imath(\star)}=k^1_\star k^2_{\imath(\star)}=0$ this implies that $c^2_{\imath(\star)}=k^2_{\imath(\star)}=0$, i.e. $B^2_{\imath(\star)}$ is invertible. Thus, $\imath(\star)=0,\star=\infty$ and $\bullet=1$. We thus conclude

- B^2_0 is invertible, B^1_0 is full-rank, B^1_1 is injective and B^1_∞ is not full-rank.
- $c^1_1=c^2_1=1$.

Since B^2_0 is invertible, we may assume that $A^2_0=D^2_0=0$. If B^1_0 is injective, we may also assume that $D^1_0=0$ and that $\mathrm{Coker}(\mathfrak{D})$ includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty)\otimes\mathrm{Coker}(B^2_\infty)$ and of size $c^1_\infty c^2_\infty$. Since $c^1_\infty\neq 0$ we conclude that B^2_∞ is surjective. From here $a^2_\infty=a^2_1\leq a^2_0-1$ and $1-k^2_1=c^2_1-k^2_1=a^2_0-a^2_\infty\geq 1$. We thus find $k^2_1=0$ and K_2 is full-rank, a contradiction. Thus $k^1_0>0$ and $c^1_0=0$. From $k^1_0k^2_\infty=0$ we find $k^2_\infty=0$, i.e. B^2_∞ is injective and the conditions of (S-1) are satisfied. A similar argument reduces the case $\widehat{k}(\mathfrak{D})=1$ and $\widehat{c}(\mathfrak{D})=0$ to (S-2). □

Proposition 6.4. *Given the pair of knots (K_1,K_2) where K_1 is full-rank and $(\circ,\bullet,*)\in\{(0,1,\infty),(1,\infty,0),(\infty,0,1)\}$,*

(K) *If $B^1_\circ,B^1_\bullet$ are injective and B^1_* is surjective then*

$$c(\mathfrak{D})\geq c^1_\bullet c^2_{\imath(\bullet)}+c^1_\circ c^2_{\imath(\circ)}\quad\textit{and}\quad k(\mathfrak{D})\geq k(X^1_\bullet)k(B^2_{\imath(*)}X^2_{\imath(\bullet)}).$$

(C) *If $B^1_\circ,B^1_\bullet$ are surjectiveand B^1_* is injective then*

$$k(\mathfrak{D})\geq k^1_\bullet k^2_{\imath(\bullet)}+k^1_\circ k^2_{\imath(\circ)}\ \textit{and}\ c(\mathfrak{D})\geq c(X^1_\bullet)c(X^2_{\imath(\bullet)}B^2_{\imath(*)}).$$

Proof. The first claim in either of cases (K) and (C) is already observed in our earlier discussions. We thus need to prove the second claim in each one of the above two cases. The proofs are very similar and we will only go through the proof for

$(\circ, \bullet, *) = (0, 1, \infty)$. In fact, the proof of claim (C) for $(\circ, \bullet, *)$ is almost identical to the proof of claim (K) for $(\imath(\bullet), \imath(\circ), \imath(*))$ because of the symmetry in the block presentation of $\mathfrak{D}$.

We assume $(\circ, \bullet, *) = (0, 1, \infty)$. In case (K), after an admissible change of basis, we may assume that $\tau_0(K_1), \tau_1(K_1)$ and $\tau_\infty(K_1)$ take the standard form of (20). Since $D_0^1 = D_1^1 = A_\infty^1 = 0$, the $(3,2)$ entry and the $(6,6)$ entry of the matrix $\mathfrak{D}$ are both the identity matrix. The matrix $\mathfrak{D}$ is thus equivalent to the matrix

$$\begin{pmatrix} \begin{matrix} D_\infty^1 B_1^1 \otimes B_1^2 A_0^2 + \\ B_1^1 A_0^1 \otimes D_\infty^2 B_1^2 \end{matrix} & B_1^1 B_0^1 \otimes I & D_\infty^1 A_1^1 \otimes B_1^2 A_0^2 & I \otimes B_1^2 B_0^2 \\ I \otimes B_\infty^2 B_1^2 & 0 & 0 & B_0^1 B_\infty^1 \otimes I \\ B_\infty^1 B_1^1 \otimes I & I \otimes B_0^2 B_\infty^2 & B_\infty^1 A_1^1 \otimes I & X_1^1 B_\infty^1 \otimes B_0^2 X_1^2 \\ D_\infty^1 B_1^1 \otimes D_1^2 A_0^2 & 0 & \begin{matrix} I \otimes I + \\ D_\infty^1 A_1^1 \otimes D_1^2 A_0^2 \end{matrix} & I \otimes D_1^2 B_0^2 \end{pmatrix}.$$

Replacing the block forms for $\tau_\star(K_1)$ gives the following presentation of the above matrix

$$\begin{pmatrix} \star & \star & I \otimes I & 0 & 0 & \star & I \otimes B_1^2 B_0^2 & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & X_1^1 \otimes I & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star \\ \star & \star & I \otimes B_0^2 B_\infty^2 & 0 & 0 & \star & X_1^1 X_1^1 \otimes B_0^2 X_1^2 & \star & \star \\ \star & \star & 0 & I \otimes I & 0 & \star & \star & \star & \star \\ \star & \star & 0 & 0 & I \otimes I & \star & 0 & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star \end{pmatrix}.$$

After subtracting $I \otimes B_0^2 B_\infty^2$ times the first row from the fifth row, the identity matrices which appear in the entries $(1,3), (7,4)$ and $(8,5)$ of the above matrix become the only non-zero entries of their respective columns. They may thus be used for the cancellation of the third, the fourth and the fifth columns against the first, the seventh and the eighth rows. We thus arrive at a 6×6 matrix equivalent to $\mathfrak{D}$, which is of the form

$$\begin{pmatrix} \star & \star & \star & 0 & \star & \star \\ \star & \star & \star & 0 & \star & \star \\ \star & \star & \star & X_1^1 \otimes I & \star & \star \\ \star & \star & \star & 0 & \star & \star \\ \star & \star & \star & (I + X_1^1 X_1^1) \otimes B_0^2 X_1^2 & \star & \star \\ \star & \star & \star & 0 & \star & \star \end{pmatrix}.$$

Since the kernel of $\mathfrak{D}$ includes a subspace which is isomorphic to the kernel corresponding to the fourth column we find $k(\mathfrak{D}) \geq k(X_1^1) k(B_0^2 X_1^2)$.

For case (C), using Lemma 5.6 choose the standard block form of (21) for K_1. In particular, A_0^1, A_1^1 and D_∞^1 are all zero. The entries $(3,2)$ and $(5,4)$ of $\mathfrak{D}$ are thus identity matrices which may be used for cancellation. Add $B_\infty^1 B_1^1 \otimes B_0^2 X_1^2$ times the second row of the resulting matrix to its third row, add $B_\infty^1 B_1^1 \otimes D_0^2 X_1^2$ times the second row to the last row, and note that $B_1^1 D_1^1 = 0$ to arrive at the following matrix, which is equivalent to $\mathfrak{D}$:

$$\begin{pmatrix} 0 & B_1^1 B_0^1 \otimes I & I \otimes B_1^2 B_0^2 & 0 \\ I \otimes B_\infty^2 B_1^2 & D_1^1 B_0^1 \otimes B_\infty^2 A_1^2 & B_0^1 B_\infty^1 \otimes I & B_0^1 A_\infty^1 \otimes I \\ B_\infty^1 B_1^1 \otimes (I + X_0^2 X_0^2) & I \otimes B_0^2 B_\infty^2 & D_0^1 B_\infty^1 \otimes B_0^2 A_\infty^2 & D_0^1 A_\infty^1 \otimes B_0^2 A_\infty^2 \\ B_\infty^1 B_1^1 \otimes D_0^2 X_1^2 B_\infty^2 B_1^2 & I \otimes D_0^2 B_\infty^2 & D_0^1 B_\infty^1 \otimes D_0^2 A_\infty^2 & \begin{matrix} I \otimes I + \\ D_0^1 A_\infty^1 \otimes D_0^2 A_\infty^2 \end{matrix} \end{pmatrix}.$$

Replacing the block forms of (21) for $\tau_0(K_1),\tau_1(K_1)$ and $\tau_\infty(K_1)$ we arrive at a matrix of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & I\otimes I & I\otimes B_1^2B_0^2 & 0 & 0 & 0\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
0 & X_0^1\otimes(I+X_0^2X_0^2) & 0 & 0 & I\otimes B_0^2B_\infty^2 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star
\end{pmatrix},$$

which is in turn equivalent to a matrix of the form

$$\begin{pmatrix}
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
0 & X_0^1\otimes(I+X_0^2X_0^2) & 0 & 0 & I\otimes X_\infty^2B_0^2 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star
\end{pmatrix},$$

In particular, we conclude $c(\mathfrak{D})\geq c(X_0^1)c(X_\infty^2B_0^2)$. This completes the proof of case (C) when $(\circ,\bullet,*)=(0,1,\infty)$. $\square$

6.2. **The special cases (S-1) and (S-2).**

Lemma 6.5. *If (K_1,K_2) is a special pair of type (S-1) or (S-2) then one of the knots K_1 or K_2 is trivial.*

Proof. Suppose otherwise that (K_1,K_2) is a special pair of type (S-1) and that both K_1 and K_2 are non-trivial. After an admissible change of basis, assume that

$$\tau_0^2=\left(\begin{array}{cc|cc} 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ \hline I & 0 & 0 & 0\\ 0 & I & 0 & 0\end{array}\right),\quad \tau_\infty^2=\left(\begin{array}{cc|c} 0 & 0 & I\\ 0 & \star & 0\\ \hline I & 0 & 0\end{array}\right)\quad\text{and}\quad \tau_1^2=\left(\begin{array}{c|cc} \star & X_\infty^2 & \star\\ \hline \star & \star & \star\\ \star & \star & \star\end{array}\right) \tag{22}$$

In particular, A_0^2, D_0^2 and D_∞^2 are zero. We may also assume that

$$\tau_0^1=\left(\begin{array}{c|cc} 0 & I & 0\\ \hline I & 0 & 0\\ 0 & 0 & \star\end{array}\right),\quad \tau_1^1=\left(\begin{array}{cc|c} 0 & 0 & I\\ 0 & \star & 0\\ \hline I & 0 & 0\end{array}\right)\quad\text{and}\quad \tau_\infty^1=\left(\begin{array}{cc|cc} \star & \star & X_\infty^1 & \star\\ \star & \star & \star & \star\\ \hline \star & \star & \star & \star\\ \star & \star & \star & \star\end{array}\right) \tag{23}$$

In particular, A_0^1 and D_1^1 are zero. The identity matrices which appear as entries $(3,2)$, $(5,4)$ and $(6,6)$ in $\mathfrak{D}(K_1,K_2)$ may be used for cancellation to obtain the equivalent matrix

$$\begin{pmatrix}
0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2\\
I\otimes B_\infty^2B_1^2 & 0 & B_0^1B_\infty^1\otimes I\\
B_\infty^1B_1^1\otimes I & I\otimes B_0^2B_\infty^2 & \begin{array}{c} D_0^1B_\infty^1\otimes B_0^2A_\infty^2\\ +X_1^1B_\infty^1\otimes B_0^2X_1^2\\ +B_\infty^1A_1^1\otimes D_1^2B_0^2\end{array}
\end{pmatrix}.$$

Subtracting $X_1^1B_\infty^1\otimes B_0^2B_\infty^2$ times the first row from the third row we arrive at the equivalent matrix

$$\begin{pmatrix}
0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2\\
I\otimes B_\infty^2B_1^2 & 0 & B_0^1B_\infty^1\otimes I\\
B_\infty^1B_1^1\otimes I & (I+X_1^1X_1^1)\otimes B_0^2B_\infty^2 & \begin{array}{c} D_0^1B_\infty^1\otimes B_0^2A_\infty^2\\ +B_\infty^1A_1^1\otimes D_1^2B_0^2\end{array}
\end{pmatrix}.$$

Replacing the block forms of (22) and (23), the above matrix takes the form

$$\begin{pmatrix} 0 & \star & I\otimes I & 0 & I\otimes X^2_\infty & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ I\otimes X^2_\infty & \star & 0 & 0 & X^1_\infty\otimes I & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ X^1_\infty\otimes I & \star & (I+X^1_\infty X^1_\infty)\otimes I & 0 & 0 & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ \star & \star & \star & I\otimes I & 0 & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \end{pmatrix}.$$

Subtract $(I+X^1_\infty X^1_\infty)\otimes I$ times the first row from the fifth row and use the identity matrices which appear as $(1,3)$ and $(7,4)$ entries of the above matrix for cancellation to arrive at the following equivalent matrix

$$\begin{pmatrix} 0 & \star & 0 & \star & \star & \star \\ I\otimes X^2_\infty & \star & X^1_\infty\otimes I & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \\ X^1_\infty\otimes I & \star & (I+X^1_\infty X^1_\infty)\otimes X^2_\infty & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \end{pmatrix}.$$

From the above presentation we conclude

$$k(\mathfrak{D}) \geq 2k(X^1_\infty)k(X^2_\infty) \geq 2.$$

This contradiction rules out the case (S-1). Ruling out the case (S-2) is similar. □

7. Incompressible tori in homology spheres

7.1. **The main theorem.**

Theorem 7.1. *Suppose that K_i is a non-trivial knot in the homology sphere Y_i for $i=1,2$. Let $Y=Y(K_1,K_2)$ denote the three-manifold obtained by splicing the complements of K_1 and K_2. Then the rank of $\widehat{\mathrm{HF}}(Y)$ is bigger than one.*

Proof. Suppose otherwise that Y is a L-space. Thus (K_1,K_2) is a special pair. By Proposition 6.3 and Lemma 6.5 we may assume that K_1 is full-rank. In particular, one of the cases (K) or (C) from Proposition 6.4 will happen. Note that in case (K) the kernel of $\mathfrak{D}$ is necessarily non-trivial by Lemma 5.4, while in case (C) the cokernel of $\mathfrak{D}$ is non-trivial.
Let us assume that (K) is the case. Thus $c(\mathfrak{D})=0$ and $k(X^1_\bullet)=k(B^2_{\imath(*)}X^2_{\imath(\bullet)})=1$. Note that $\mathrm{Ker}(B^2_{\imath(*)})\subset \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$, which implies that either $B^2_{\imath(*)}$ is injective or $\mathrm{Ker}(B^2_{\imath(*)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$. If the latter happens, we find

$$\mathrm{Ker}(B^2_{\imath(*)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)}X^2_{\imath(\bullet)})=\cdots=\mathrm{Ker}(0),$$

since $X^2_{\imath(\bullet)}$ is nilpotent by Lemma 5.4. Since $B^2_{\imath(*)}\neq 0$ this can not happen and we conclude that $B^2_{\imath(*)}$ is injective.
Let us first assume that B^1_0 is not invertible. Then $c^1_\bullet, c^1_\circ \neq 0$. Since $c^1_\bullet c^2_{\imath(\bullet)} = c^1_\circ c^2_{\imath(\circ)}=0$ we conclude that $B^2_{\imath(\circ)}$ and $B^2_{\imath(\bullet)}$ are both surjective. Thus K_2 is full-rank and by part (C) of Proposition 6.4 $c(\mathfrak{D})>0$. This contradiction implies that B^1_0 is invertible. Moreover, the argument implies that $0\in\{\circ,\bullet\}$ and at least one of $c^2_{\imath(\circ)}$ and $c^2_{\imath(\bullet)}$ is trivial. It is easy to conclude from here that we are then either in case (S-1) or case (S-2) of Proposition 6.3, which are both excluded by Lemma 6.5. The contradiction rules out case (K) of Proposition 6.4. Excluding the case (C) is completely similar. □

Corollary 7.2. *If the homology sphere Y contains an incompressible torus then* $\mathrm{rnk}(\widehat{\mathrm{HF}}(Y,\mathbb{F}) > 1$.

Proof. If Y contains an incompressible torus T, T will be separating and there will be a pair of curves λ and μ on T such that λ is homologically trivial on one side of T and μ is homologically trivial on the other side of T. Since Y is a homology sphere, the intersection number of μ and λ is one. Let U_1 and U_2 be the two components of $Y-T$ and let U_1 be the component containing a surface which bounds λ. Capping off $\mu \subset T = \partial U_1$ by a disk and then gluing a three-ball gives a three-manifold Y_1. The simple closed curve λ represents a knot $K_1 \subset Y_1$. Similarly capping off $\lambda \subset T = \partial U_2$ by a disk and then gluing a three-ball gives a three-manifold Y_2 and μ represents a knot $K_2 \subset Y_2$. Both Y_1 and Y_2 are homology spheres and Y is obtained by splicing K_1 and K_2. Since T is incompressible, both K_1 and K_2 are non-trivial and Theorem 7.1 completes the proof of this corollary. $\square$

7.2. **Applications.** We may use the relation between Khovanov homology of a knot inside the standard sphere and the Heegaard Floer homology of its branched double-cover, discovered by Ozsváth and Szabó [OS4], to show the non-triviality of Khovanov homology for certain classes of knots. We emphasize again that the results presented here are all special cases of the the theorem of Kronheimer and Mrowka [KM] that Khovanov homology is an unknot detector.

Definition 7.3. *A prime knot $K \subset S^3$ is an n-*string composite *if there is an embedded 2-sphere intersecting the knot transversely which separates (S^3, K) into prime n-string tangles. A 2-string composite knot is called a* doubly composite knot.

We refer the reader to [Blei] for more on doubly composite and doubly prime knots, and only quote the following lemma from that paper:

Lemma 7.4. *A prime knot $K \subset S^3$ is a doubly composite knot if and only if the double cover $\Sigma(K)$ of S^3 branched over the knot K contains an incompressible torus T which is invariant under the non-trivial covering translation and meets the fixed point set of this map precisely in 4 points, and separates $\Sigma(K)$ into irreducible boundary irreducible pieces.*

Corollary 7.5. *If the prime knot $K \subset S^3$ is doubly composite, the rank of its reduced Khovanov homology group $\widetilde{\mathrm{Kh}}(K)$ is bigger than* 1.

Proof. If K is doubly composite, by Lemma 7.4 there exists an incompressible torus T inside the three-manifold $\Sigma(K)$. Thus the rank of $\widehat{\mathrm{HF}}(\Sigma(K),\mathbb{F})$ is bigger than 1. By the main theorem of [OS4] there is a spectral sequence whose E^2-term consists of Khovanov's reduced homology $\widetilde{\mathrm{Kh}}(K)$ of the mirror of K with coefficients in $\mathbb{F}$ which converges to $\widehat{\mathrm{HF}}(\Sigma(K),\mathbb{F})$, and is of rank greater than 1 by Theorem 7.1. Thus the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1 as well. $\square$

Furthermore, if K is a prime satellite knot, we will have an incompressible torus in the complement of K. This torus gives an incompressible torus in the double cover $\Sigma(K)$ of S^3 branched over the knot K. Thus, Heegaard Floer homology of $\Sigma(K)$ will be non-trivial. We thus have the following corollary:

Corollary 7.6. *If $K \subset S^3$ is a prime satellite knot the rank of its reduced Khovanov homology group $\widetilde{\mathrm{Kh}}(K)$ is greater than* 1.

In fact, we may prove a slightly more general statement:

Proposition 7.7. *If the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ of a non-trivial knot $K \subset S^3$ is one, the double cover $\Sigma(K)$ of S^3, branched over the knot K, is hyperbolic.*

Proof. Note that if a knot K is doubly composite Corollary 7.5 implied that the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1. Thus, K has to be doubly prime. By Thurston's orbifold geometrization theorem (see [BP] and [CHK]) the branched double cover $\Sigma(K)$ is a geometric manifold and there are three possible cases.
1- $\Sigma(K)$ is a Lens space and thus admits a spherical structure. If $\widehat{\mathrm{HF}}(\Sigma(K))$ is one dimensional, $\Sigma(K)$ is forced to be the standard sphere and K is trivial. Thus in this case, the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1 only if K is trivial.
2- $\Sigma(K)$ admits a Seifert fibration and K is a Montesinos knot with at most three rational tangles. If $\Sigma(K)$ is not a homology sphere, $\widetilde{\mathrm{Kh}}(K)$ is clearly different from $\mathbb{F}$, and if it is a homology sphere which admits a Seifert fibration and $\widehat{\mathrm{HF}}(\Sigma(K)) = \mathbb{F}$, we know (see [Rus] or [Ef5]) that $\Sigma(K)$ is either the standard sphere, or the Poincaré sphere. Moreover, for $\Sigma(K)$ to be the Poincaré sphere we should have $K = T(3,5)$, i.e. K is the $(3,5)$-torus knot, or equivalently $(-2,3,5)$-pretzel knot, which is 10_{124} in Rolfsen's table (see [HW] and [Rolf]). $\widetilde{\mathrm{Kh}}(T(3,5))$ has rank 7 by direct computation [Shu].
3- $\Sigma(K)$ admits a hyperbolic structure which is invariant under the deck transformation.
Having ruled out the first two possibilities, the proof is complete. □

The knots K with the property that $\Sigma(K)$ admits a hyperbolic structure which is invariant under the involution of $\Sigma(K)$ are called π-*hyperbolic*. The hyperbolic structure comes from a hyperbolic structure on $S^3 - K$ which becomes a singular folding with angle π around K. Thus in particular, π-hyperbolic knots are hyperbolic.

Corollary 7.8. *Assuming Conjecture 1.2, if the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ for a knot $K \subset S^3$ is equal to $\mathbb{F}$, K is the unknot.*

Proof. Suppose K is not the unknot. By Proposition 7.7, if $\widetilde{\mathrm{Kh}}(K) = \mathbb{F}$, the branched double cover $\Sigma(K)$ is hyperbolic. Conjecture 1.2 then implies that $\widehat{\mathrm{HF}}(\Sigma(K))$ is non-trivial, and by the correspondence of [OS4],

$$1 = \mathrm{rnk}\big(\widetilde{\mathrm{Kh}}(K)\big) \geq \mathrm{rnk}(\widehat{\mathrm{HF}}(\Sigma(K))) > 1.$$

This contradiction implies that $\widetilde{\mathrm{Kh}}(K) \neq \mathbb{F}$. □

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